J Stat Phys (2008) 131: 415–444
DOI 10.1007/s10955-007-9477-8

Analysis of Equilibrium States of Markov Solutions to the 3D Navier-Stokes Equations Driven by Additive Noise

Marco Romito

Received: 20 September 2007 / Accepted: 20 December 2007 / Published online: 7 February 2008

Abstract We prove that every Markov solution to the three dimensional Navier-Stokes equations with periodic boundary conditions driven by additive Gaussian noise is uniquely ergodic. The convergence to the (unique) invariant measure is exponentially fast.

Moreover, we give a well-posedness criterion for the equations in terms of invariant measures. We also analyse the energy balance and identify the term which ensures equality in the balance.

Keywords Stochastic Navier-Stokes equations · Martingale problem · Markov selections · Invariant measures · Ergodicity · Energy balance

1 Introduction

The Navier-Stokes equations on the torus with periodic boundary conditions forced by additive Gaussian noise are a reasonable model for the analysis of homogeneous isotropic turbulence for an incompressible Newtonian fluid,

$$\begin{cases} \dot{u} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \dot{\eta}, \\ \operatorname{div} u = 0. \end{cases} \tag{1.1}$$

The equations share with their deterministic counterpart the well-known problems of well-posedness. It is reasonable, and possibly useful, to focus on special classes of solutions, having additional properties.

This paper completes the analysis developed in [10–12] (see also [1]). In these papers it was proved the existence of a Markov process which solves the equations. Moreover, under some regularity and non-degeneracy assumptions on the covariance of the driving noise, it has been shown that the associated Markov transition kernel is continuous in a space $\mathcal{W}$ with a stronger topology (than the topology of energy, namely L^2) for initial conditions in $\mathcal{W}$.

M. Romito (✉)
Dipartimento di Matematica, Università di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy
e-mail: romito@math.unifi.it

In this paper we show that, under suitable regularity assumptions on the covariance, *every* Markov solution admits an invariant measure. Moreover, if the noise is non-degenerate, the invariant measure is unique and the convergence to the (unique) invariant measure is exponentially fast.

We stress that similar results have been already obtained by Da Prato & Debussche [2], Debussche and Odasso [5] and Odasso [18], for solutions obtained as limits of spectral Galerkin approximations to (1.1), and constructed via the Kolmogorov equation associated to the diffusion. The main improvement of our results is that such conclusions are *generically* valid for *all* Markov solutions and not restricted to solutions limit to the Galerkin approximations (this would not make any difference whenever the problem is well-posed, though) and is general enough to be applied to different problems (see for instance [1]). Our analysis is essentially based on the energy balance (see Definition 2.4 and Remark 2.5), and in turn shows that such balance is the main and crucial ingredient.

It is worth noticing that the uniquely ergodic results hold for any Markov solution, hence different Markov solutions have their own (unique) invariant measure. Well-posedness of (1.1) would ensure that the invariant measure is unique. We prove that the latter condition is also sufficient, as if only one invariant measure is shared among all Markov solutions, then the problem is well-posed.

Finally, we analyse the energy balance for both the process solution to the equations and the invariant measure. Due to the lack of regularity of trajectories, the energy balance is indeed an inequality. We identify the missing term and, under the invariant measure, we relate it to the *energy flux* through wave-numbers. According to both the physical and mathematical understanding of the equations, this term *should* be zero.

A non-zero compensating term from one side would invalidate the equations as a model for phenomenological theories of turbulence, and from the other side would show that blow-up is typically true. We stress that neither the former nor the latter statements are proved here.

1.1 Details on results

In the rest of the paper we consider the following abstract version of the stochastic Navier-Stokes equations (1.1) above,

$$du + \nu Au + B(u,u) = \mathcal{Q}^{\frac{1}{2}} dW, \tag{1.2}$$

where A is the Laplace operator on the three-dimensional torus $\mathbb{T}_3$ with periodic boundary conditions and B is the projection onto the space of divergence-free vector fields with finite energy of the Navier-Stokes non-linearity (see Sect. 2.1 for more details). Moreover, W is a cylindrical Wiener process on H and $\mathcal{Q}$ is its covariance operator. We assume that $\mathcal{Q}$ is a symmetric positive operator. We shall need additional assumptions on the covariance, as the results contained in the paper holds under slightly different conditions. Here we gather the different additional assumptions we shall use.

Assumption 1.1 *The following assumptions will be used* (*one at the time*) *throughout the paper.*

[A1] *$\mathcal{Q}$ has finite trace on H.*
[A2] *there is $\alpha_0 > 0$ such that $A^{\frac{3}{4}+\alpha_0} Q^{\frac{1}{2}}$ is a bounded operator on H.*
[A3] *there is $\alpha_0 > \frac{1}{6}$ such that $A^{\frac{3}{4}+\alpha_0} Q^{\frac{1}{2}}$ is a bounded operator on H.*

[A4] *there is* $\alpha_0 > \frac{1}{6}$ *such that* $A^{\frac{3}{4}+\alpha_0} Q^{\frac{1}{2}}$ *is an invertible bounded operator on* H, *with bounded inverse.*

Notice that each of the above conditions implies the following one. We shall make clear at every stage of the paper which assumption is used.

The first main result of the paper concerns the long time behavior of solutions to (1.2). We show that *every* Markov solution is uniquely ergodic and strongly mixing (Theorem 3.1 and Corollary 3.2). Moreover, under an additional technical condition (see Remark 2.5) we prove that the convergence to the (unique) invariant measure is exponentially fast (Theorem 3.3).

We stress that uniqueness of invariant measure is relative to the Markov solution it arises from. As we do not know if the martingale problem associated to equations (1.2) is well-posed, in principle there are plenty of Markov solutions, and so plenty of invariant measures. In Sect. 4 we study a few properties of the set of invariant measures. In particular, we show the converse of the above statement, that is if there is only one common invariant measure for all Markov solutions, then the martingale problem is well posed (Theorem 4.6).

We also give some remarks on symmetries for the invariant measures (such as translations-invariance). Finally, we analyse the energy inequality (given as [M3] and [M4] in Definition 2.4, see also Remark 2.5). In particular, we identify the missing term in the inequality which, once added, provides the equality. For an invariant measure μ, we show that

$$\nu\varepsilon(\mu) + \iota(\mu) = \frac{1}{2}\sigma^2,$$

where $\frac{1}{2}\sigma^2$ is the rate of energy injected by the external force, $\varepsilon(\mu) = \mathbb{E}^\mu[|\nabla x|^2]$ is the *mean rate of energy dissipation* and $\iota(\mu)$ is the *mean rate of inertial energy dissipation*. We show also that $\iota(\mu)$ is given in terms of the *energy flux* through wave-numbers (see Frisch [14]) as

$$\iota(\mu) = \lim_{K\uparrow\infty} \mathbb{E}^\mu\Bigg[\sum_{\substack{l+m=k\\ |k|_\infty \le K,\\ |m|_\infty > K}} (x_m \cdot \overline{x_k})(m \cdot x_l)\Bigg].$$

2 Notations and Previous Results

2.1 Notations

Let $\mathbb{T}_3 = [0, 2\pi]^3$ and let $\mathcal{D}^\infty$ be the space of infinitely differentiable vector fields $\varphi : \mathbf{R}^3 \to \mathbf{R}^3$ that are divergence-free, periodic and

$$\int_{\mathbb{T}_3} \varphi(x)\, dx = 0.$$

We denote by H the closure of $\mathcal{D}^\infty$ in the norm of $L^2(\mathbb{T}_3, \mathbf{R}^3)$, and similarly by V the closure in the norm of $H^1(\mathbb{T}_3, \mathbf{R}^3)$. Let $D(A)$ be the set of all $u \in H$ such that $\Delta u \in H$ and define the *Stokes* operator $A : D(A) \to H$ as $Au = -\Delta u$. By properly identifying dual spaces, we have that $D(A) \subset V \subset H = H' \subset V' \subset D(A)'$.

The bi-linear operator $B : V \times V \to V'$ is defined as

$$\langle B(u, v), w\rangle = \sum_{i,j=1}^{3} \int_{\mathbb{T}_3} w_i(x) u_j(x) \frac{\partial v_i(x)}{\partial x_j}\, dx$$

(see Temam [21] for a more detailed account of all these notations).

Since A is a linear positive and self-adjoint operator with compact inverse, we can define powers of A. We define two hierarchies of spaces related to the problem, using powers of A. The first is given by the following spaces of *mild* regularity (since they are larger than the space V),

$$\mathcal{V}_\varepsilon = D(A^{\frac{1}{4}+\varepsilon}), \quad \varepsilon \in \left(0, \frac{1}{4}\right], \tag{2.1}$$

while the second is given by the following spaces of *strong* regularity,

$$\mathcal{W}_\alpha = D(A^{\theta(\alpha)}), \quad \alpha \in (0, \infty), \tag{2.2}$$

where θ is defined as

$$\theta(\alpha) = \begin{cases} \frac{\alpha+1}{2}, & \alpha \in (0, \frac{1}{2}), \\ \alpha + \frac{1}{4}, & \alpha \geq \frac{1}{2}. \end{cases} \tag{2.3}$$

Notice that for every ε_0 and α_0 as above,

$$\mathcal{W}_{\alpha_0} \subset \mathcal{W}_0 = V = \mathcal{V}_{\frac{1}{4}} \subset \mathcal{V}_{\varepsilon_0}.$$

In the proof of most of the results of the paper we shall use repeatedly the following inequalities.

Lemma 2.1 (Temam [21, Lemma 2.1, Part I]) *If $u \in D(A^{\alpha_1})$, $v \in D(A^{\alpha_2})$ and $w \in D(A^{\alpha_3})$, then there is a constant $c_0 = c_0(\alpha_1, \alpha_2, \alpha_3)$ such that*

$$\langle B(u, v), w\rangle_H \leq c_0 |A^{\alpha_1} u| \cdot |A^{\alpha_2+\frac{1}{2}} v| \cdot |A^{\alpha_3} w|,$$

where $\alpha_i \geq 0$ and $\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{3}{4}$ if $\alpha_i \neq \frac{3}{4}$ for all $i = 1, 2, 3$, and $\alpha_1 + \alpha_2 + \alpha_3 > \frac{3}{4}$ otherwise.

Lemma 2.2 ([11, Lemma D.2]) *Let $\alpha > 0$ and $u, v \in D(A^{\theta(\alpha)})$. If $\alpha \neq \frac{1}{2}$, there is a constant $C_0 = C_0(\alpha)$ such that*

$$|A^{\alpha-\frac{1}{4}} B(u, v)|_H \leq C_0 |A^{\theta(\alpha)} u|_H |A^{\theta(\alpha)} v|_H,$$

where θ is the map defined in (2.3). If $\alpha = \frac{1}{2}$, then B maps $D(A^{\frac{3}{4}}) \times D(A^{\frac{3}{4}})$ continuously to $D(A^{\frac{1}{4}-\varepsilon})$, for every $\varepsilon > 0$.

2.2 Markov Solutions to the Navier-Stokes Equations

In this section we recall a few definitions and result from papers [11] and [12], with some additional remarks.

2.2.1 Almost Sure Super-Martingales

We say that a process $\theta = (\theta_t)_{t\geq 0}$ on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$, adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is an *a. s. super-martingale* if it is $\mathbb{P}$-integrable and there is a set $T \subset (0,\infty)$ of null Lebesgue measure (that we call the set of *exceptional times* of θ) such that

$$\mathbb{E}[\theta_t|\mathcal{F}_s] \leq \theta_s, \tag{2.4}$$

for all $s \notin T$ and all $t > s$.

Lemma 2.3 *If $\theta = (\theta_t)_{t\geq 0}$ is an a. s. super-martingale, then for every $s \geq 0$ and every $\varphi \in C_c^\infty(\mathbf{R})$ with $\varphi \geq 0$ and* $\operatorname{Supp}\varphi \subset [s,\infty)$,

$$\mathbb{E}\left[\int \varphi'(r)\theta_r\,dr\,\middle|\,\mathcal{F}_s\right] \geq 0. \tag{2.5}$$

Proof Fix $s \geq 0$ and consider a positive smooth map φ with compact support in $[s,\infty)$. By a change of variable, using the a. s. super-martingale property,

$$\mathbb{E}\left[\frac{1}{\varepsilon}\int(\varphi(r)-\varphi(r-\varepsilon))\theta_r\,dr\,\middle|\,\mathcal{F}_s\right] = \frac{1}{\varepsilon}\mathbb{E}\left[\int_s^\infty \varphi(r)(\theta_r-\theta_{r+\varepsilon})\,dr\,\middle|\,\mathcal{F}_s\right] \geq 0,$$

and in the limit as $\varepsilon \downarrow 0$ we get (2.5). □

It is easy to see that the converse is true (that is, if (2.5) holds, then the process is an a. s. super-martingale) under the assumption that the σ-fields $\{\mathcal{F}_t : t \geq 0\}$ are countably generated and θ is lower semi-continuous (see [13]).

2.2.2 Weak Martingale Solutions

Let $\Omega = C([0,\infty); D(A)')$, let $\mathcal{B}$ be the Borel σ-field on Ω and let $\xi : \Omega \to D(A)'$ be the canonical process on Ω (that is, $\xi_t(\omega) = \omega(t)$). A filtration can be defined on $\mathcal{B}$ as $\mathcal{B}_t = \sigma(\xi_s : 0 \leq s \leq t)$.

Definition 2.4 Given $\mu_0 \in \Pr(H)$, a probability P on $(\Omega, \mathcal{B})$ is a solution starting at μ_0 to the martingale problem associated to the Navier-Stokes equations (1.2) if

[M1] $P[L^\infty_{\text{loc}}([0,\infty); H) \cap L^2_{\text{loc}}([0,\infty); V)] = 1$;

[M2] for each $\varphi \in \mathcal{D}^\infty$ the process M_t^φ, defined P-a. s. on $(\Omega, \mathcal{B})$ as

$$M_t^\varphi = \langle \xi_t - \xi_0, \varphi\rangle_H + \nu\int_0^t \langle \xi_s, A\varphi\rangle_H\,ds - \int_0^t \langle B(\xi_s,\varphi), \xi_s\rangle_H\,ds$$

is square integrable and $(M_t^\varphi, \mathcal{B}_t, P)$ is a continuous martingale with quadratic variation $[M^\varphi]_t = t|\mathcal{Q}^{\frac{1}{2}}\varphi|_H^2$;

[M3] the process $\mathcal{E}_t^1$, defined P-a. s. on $(\Omega, \mathcal{B})$ as

$$\mathcal{E}_t^1 = |\xi_t|_H^2 + 2\nu\int_0^t |\xi_s|_V^2\,ds - t\operatorname{Tr}[\mathcal{Q}]$$

is P-integrable and $(\mathcal{E}_t^1, \mathcal{B}_t, P)$ is an a. s. super-martingale;

[M4] for each $n \geq 2$, the process $\mathcal{E}_t^n$, defined P–a. s. on $(\Omega, \mathcal{B})$ as

$$\mathcal{E}_t^n = |\xi_t|_H^{2n} + 2n\nu \int_0^t |\xi_s|_H^{2n-2} |\xi_s|_V^2 \, ds - n(2n-1) \operatorname{Tr}[\mathcal{Q}] \int_0^t |\xi_s|_H^{2n-2} \, ds$$

is P-integrable and $(\mathcal{E}_t^n, \mathcal{B}_t, P)$ is an a. s. super-martingale;

[M5] μ_0 is the marginal of P at time $t = 0$.

Remark 2.5 (Enhanced martingale solutions) A slightly different approach has been followed in [1] to show existence of Markov solution for a different model (an equation for surface growth driven by space-time white noise), as the energy balance has been given in terms of an almost sure property. In the Navier-Stokes setting of this paper the property reads (some equivalent statements are possible as in [1])

[M3-as] there is a set $T_{P_x} \subset (0, \infty)$ of null Lebesgue measure such that for all $s \notin T_{P_x}$ and all $t \geq s$,

$$P_x[\mathcal{G}_t(v, z) \leq \mathcal{G}_s(v, z)] = 1,$$

where $\mathcal{G}$ is defined as

$$\mathcal{G}_t(v, z) = \frac{1}{2}|v_t|_H^2 + \nu \int_0^t |v_r|_V^2 \, dr + \int_0^t \langle v_r, B(v_r + z_r, z_r)\rangle_H \, dr,$$

z is the solution to the Stokes problem (A.2) and $v = \xi - z$. It is possible to show that, as in [1], there exist Markov solutions which additionally satisfy [M3-as]. We shall assume this statement (see [13] for more details).

2.3 Previous Results

In the next theorems we summarize some results on existence and regularity of Markov solutions to the Navier-Stokes equations (1.2). First we show that there is a Markov solution to the Navier-Stokes equations (1.2).

Theorem 2.6 ([11, Theorem 4.1]) *Under condition* [A1] *of Assumption* 1.1, *there exists a family* $(P_x)_{x\in H}$ *of weak martingale solutions* (*as defined above in Definition* 2.4), *with* P_x *starting at the measure concentrated on* x, *for each* $x \in H$, *and the* almost sure Markov property *holds. More precisely, for every* $x \in H$ *there is a set* $T \subset (0, \infty)$ *of null Lebesgue measure such that for all* $s \notin T$, *all* $t \geq s$ *and all bounded measurable* $\phi : H \to \mathbf{R}$,

$$\mathbb{E}^{P_x}[\phi(\xi_t)|\mathcal{B}_s] = \mathbb{E}^{P_{\xi_s}}[\phi(\xi_{t-s})].$$

The map $x \mapsto P_x$ is in principle, from the above result, only measurable. The regularity of dependence from initial condition can be significantly improved under stronger assumptions on the noise, as shown by the theorem below.

If $(P_x)_{x\in H}$ is a Markov solution, the transition semigroup[1] associated to the solution is defined as

$$\mathcal{P}_t\varphi(x) = \mathbb{E}^{P_x}[\varphi(\xi_t)], \quad x \in H, \ t \geq 0, \tag{2.6}$$

[1] Notice that, due to the Markov property holding only almost surely, the family of operators $(\mathcal{P}_t)_{t\geq 0}$ is not a semigroup, as the semigroup property holds for almost every time.

for every bounded measurable $\varphi : H \to \mathbf{R}$.

Theorem 2.7 ([11, Theorem 5.11]) *Under condition* [A4] *of Assumption* 1.1, *the transition semigroup* $(\mathcal{P}_t)_{t\geq 0}$ *associated to* every *Markov solution* $(P_x)_{x\in H}$ *is strong Feller in the topology of* $\mathcal{W}_{\alpha_0}$. *More precisely,* $\mathcal{P}_t\phi \in C_b(\mathcal{W}_{\alpha_0})$ *for every* $t > 0$ *and every bounded measurable* $\phi : H \to \mathbf{R}$.

The regularity result can be given more explicitly in terms of quasi-Lipschitz regularity (that is, Lipschitz up to a logarithmic correction) as in [12], albeit the estimate given there holds true only for $\alpha_0 = \frac{3}{4}$ (an extension to all values of $\alpha_0 > \frac{1}{6}$ can be found in [13]).

3 Existence and Uniqueness of the Invariant Measure

In this section we prove existence of invariant measures by means of the classical Krylov-Bogoliubov method. Let $(P_x)_{x\in H}$ be a Markov solution and denote by $(\mathcal{P}_t)_{t\geq 0}$ its transition semigroup (see (2.6)). Let $x_0 \in H$ and

$$\mu_t = \frac{1}{t}\int_0^t \mathcal{P}_s^* \delta_{x_0}, \tag{3.1}$$

where δ_{x_0} is the Dirac measure concentrated on x_0. It is known (see for example Da Prato & Zabczyk [3]) that any limit point of the family of probability measures $(\mu_t)_{t\geq 0}$ is an invariant measure for $(\mathcal{P}_t)_{t\geq 0}$, provided that the family is tight in the topology where the transition semigroup is Feller.

Theorem 3.1 *Assume* [A2] *of Assumption* 1.1. *Let* $(P_x)_{x\in H}$ *be any Markov solution to the Navier-Stokes equations* (*see Theorem* 2.6) *and let* $(\mathcal{P}_t)_{t\geq 0}$ *be the associated transition semigroup. Then the family of probability measures* $(\mu_t)_{t\geq 1}$ *is tight in* $\mathcal{W}_{\alpha_0}$.

The above theorem, the strong Feller property ensured by Theorem 2.7 and Doob's theorem (see Da Prato & Zabczyk [4]) immediately imply the following result.

Corollary 3.2 *Under* [A4] *of Assumption* 1.1, every *Markov selection to the Navier-Stokes equations has a unique invariant measure* $\mu_\star$, *which is strongly mixing. Moreover, there are* $\delta > 0$ *and* $\gamma > 0$ (*depending only on* α_0) *such that*

$$\mathbb{E}^{\mu_\star}[|A^\delta x|_{\mathcal{W}_{\alpha_0}}^\gamma] < \infty.$$

The convergence of transition probabilities to the unique invariant measure can be further improved if, under the same assumptions of above results, we deal with the *enhanced martingale solutions* introduced in Remark 2.5. This is a technical requirement that makes the proof of Theorem 3.3 below simple and, above all, feasible.

Theorem 3.3 *Assume* [A4] *of Assumption* 1.1 *and consider an arbitrary Markov solution* $(P_x)_{x\in H}$ *made of enhanced martingale solutions* (*see Remark* 2.5). *Let* $\mu_\star$ *be its unique invariant measure. Then there are constants* $C_{exp} > 0$ *and* $a > 0$ (*independent of the Markov solution and depending only on the data of problem*) *such that*

$$\|\mathcal{P}_t^*\delta_{x_0} - \mu\|_{\mathsf{TV}} \leq C_{exp}(1 + |x_0|_H^2)\mathrm{e}^{-at},$$

for all $t > 0$ and $x_0 \in H$, where $\|\cdot\|_{\mathrm{TV}}$ is the total variation distance on measures.

Remark 3.4 The proof of Theorem 3.3 above actually shows a slightly stronger convergence, namely

$$\sup_{\|\phi\|_V \le 1} |\mathcal{P}_t\phi(x_0) - \int \phi(x)\,\mu(dx)| \le C_{exp}(1 + |x|_H^2)\mathrm{e}^{-at}$$

for every $x \in H$ and $t \ge 0$, with same constants C_{exp} and a, where the norm $\|\cdot\|_V$ is defined on Borel measurable maps $\phi : H \to \mathbf{R}$ as

$$\|\phi\|_V = \sup_{x\in H} \frac{|\phi(x)|}{1 + |x|_H^2}$$

(see Goldys & Maslowski [15] for details).

From Theorem 13 of [12] and again from Theorem 4.2.1 of Da Prato & Zabczyk [4] we also deduce the following result.

Corollary 3.5 *Under the assumptions of previous corollary, let μ_1 and μ_2 be the invariant measures associated to two different Markov selections. Then the two measures are mutually equivalent.*

The rest of the section is devoted to the proof of Theorems 3.1 and 3.3.

3.1 The Proof of Theorem 3.1

We fix a Markov solution $(P_x)_{x\in H}$. Prior to the proof of the theorem, we show two lemmas on momenta of the solution. The second lemma is the crucial one.

Lemma 3.6 *Assume* [A1] *of Assumption 1.1. Then for every $x \in H$ and $t \ge 0$,*

$$\mathbb{E}^{P_x}[|\xi_t|_H^2] \le |x|_H^2 \mathrm{e}^{-2\nu t} + \frac{\sigma^2}{2\nu}(1 - \mathrm{e}^{-2\nu t}).$$

Proof The result easily follows from the super-martingale property [M3], Poincaré inequality and Gronwall's lemma (see for example [19] for details). □

Lemma 3.7 *Assume* [A2] *of Assumption 1.1. Then there are $C > 0$, $\delta > 0$ and $\gamma > 0$ depending only on ε_0, α_0, ν and σ^2 (but not on the Markov solution) such that for $x_0 \in H$ and $t \ge 1$,*

$$\mathbb{E}^{P_{x_0}}\left[\frac{1}{t}\int_0^t |A^\delta \xi_s|_{W_{\alpha_0}}^{2\gamma}\,ds\right] \le C(1 + |x_0|_H^2).$$

A slight modification of the argument in the proof below provides an inequality similar to that of the lemma also for $t < 1$.

Proof Let $\varepsilon_0 \in (0, \frac{1}{4}]$ with $\varepsilon_0 < 2\alpha_0$. We first prove the statement of the lemma for $x_0 \in \mathcal{V}_{\varepsilon_0}$.

Consider values $\delta = \delta(\varepsilon_0, \alpha_0)$, $\gamma = \gamma(\varepsilon_0, \alpha_0)$ provided by Theorem A.2. For every fixed value $M > 0$ we choose $R \ge 1 + 2|x_0|_H^2$, whose value will be given explicitly later, and we denote by ε_R the small time where the blow-up estimate (A.5) of Theorem A.1 holds true.

Fix $t \geq 1$ and $\varepsilon \leq \varepsilon_R$, and let $n_\varepsilon \in \mathbf{N}$ be the largest integer such that $\varepsilon n_\varepsilon \leq t$. By the Markov property,

$$\begin{aligned}
\mu_t[|A^\delta x|^2_{\mathcal{W}_{\alpha_0}} \geq M] &= \frac{1}{t}\int_0^t P_{x_0}[|A^\delta \xi_s|^2_{\mathcal{W}_{\alpha_0}} \geq M]\,ds \\
&\leq \frac{1}{t}\sum_{k=0}^{n_\varepsilon}\int_{k\varepsilon}^{k\varepsilon+\varepsilon} P_{x_0}[|A^\delta \xi_s|^2_{\mathcal{W}_{\alpha_0}} \geq M]\,ds \\
&= \frac{1}{t}\sum_{k=0}^{n_\varepsilon}\mathbb{E}^{P_{x_0}}\left[\int_{k\varepsilon}^{k\varepsilon+\varepsilon} P_{\xi_{k\varepsilon}}[|A^\delta \xi'_{s-k\varepsilon}|^2_{\mathcal{W}_{\alpha_0}} \geq M]\,ds\right] \\
&= \frac{1}{t}\sum_{k=0}^{n_\varepsilon}\mathbb{E}^{P_{x_0}}\left[\int_0^\varepsilon P_{\xi_{k\varepsilon}}[|A^\delta \xi'_s|^2_{\mathcal{W}_{\alpha_0}} \geq M]\,ds\right],
\end{aligned} \tag{3.2}$$

where μ_t is the measure defined in (3.1). Now, by Theorem A.1, for every $x \in \mathcal{V}_{\varepsilon_0}$ such that $|x|^2_{\mathcal{V}_{\varepsilon_0}} \leq R$,

$$P_x[|A^\delta \xi_s|^2_{\mathcal{W}_{\alpha_0}} \geq M] \leq P_x^{(\varepsilon_0,R)}[|A^\delta \xi_s|^2_{\mathcal{W}_{\alpha_0}} \geq M] + P_x[\tau^{(\varepsilon_0,R)} \leq s]$$

and so, by using (A.5) and Chebychev inequality,

$$\begin{aligned}
&P_{\xi_{k\varepsilon}}[|A^\delta \xi'_s|^2_{\mathcal{W}_{\alpha_0}} \geq M] \\
&\quad\leq \big(P_{\xi_{k\varepsilon}}^{(\varepsilon_0,R)}[|A^\delta \xi'_s|^2_{\mathcal{W}_{\alpha_0}} \geq M] + P_{\xi_{k\varepsilon}}[\tau^{(\varepsilon_0,R)} \leq s]\big)\mathbb{1}_{\{|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} \leq R\}} + \mathbb{1}_{\{|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R\}} \\
&\quad\leq \mathbb{1}_{\{|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R\}} + \frac{1}{M^\gamma}\mathbb{E}^{P_{\xi_{k\varepsilon}}^{(\varepsilon_0,R)}}[|A^\delta \xi'_s|^{2\gamma}_{\mathcal{W}_{\alpha_0}}] + c_1 \mathrm{e}^{-c_2\frac{R^2}{\varepsilon_R}}.
\end{aligned}$$

We use the above inequality in (3.2) and we apply Theorem A.2 and the previous lemma,

$$\begin{aligned}
&\mu_t[|A^\delta x|^2_{\mathcal{W}_{\alpha_0}} \geq M] \\
&\quad\leq \frac{1}{t}\sum_{k=0}^{n_\varepsilon}\mathbb{E}^{P_{x_0}}\left[\varepsilon\mathbb{1}_{\{|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R\}} + \frac{1}{M^\gamma}\mathbb{E}^{P_{\xi_{k\varepsilon}}^{(\varepsilon_0,R)}}\left[\int_0^\varepsilon |A^\delta \xi'_s|^{2\gamma}_{\mathcal{W}_{\alpha_0}}\,ds\right] + c_1\varepsilon \mathrm{e}^{-c_2\frac{R^2}{\varepsilon_R}}\right]. \\
&\quad\leq \frac{1}{t}\sum_{k=0}^{n_\varepsilon}\left(\varepsilon P_{x_0}[|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R] + \frac{C}{M^\gamma}(1+\varepsilon+\mathbb{E}^{P_{x_0}}[|\xi_{k\varepsilon}|^2_H]) + c_1\varepsilon \mathrm{e}^{-c_2\frac{R^2}{\varepsilon_R}}\right). \\
&\quad\leq \frac{\varepsilon}{t}\sum_{k=0}^{n_\varepsilon} P_{x_0}[|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R] + \frac{Cn_\varepsilon}{tM^\gamma}(1+\varepsilon+|x_0|^2_H) + c_1\frac{n_\varepsilon\varepsilon}{t}\mathrm{e}^{-c_2\frac{R^2}{\varepsilon_R}} \\
&\quad\leq \frac{\varepsilon}{t}\sum_{k=0}^{n_\varepsilon} P_{x_0}[|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R] + \frac{C}{\varepsilon M^\gamma}(1+|x_0|^2_H) + c_1 \mathrm{e}^{-c_2\frac{R^2}{\varepsilon_R}}.
\end{aligned}$$

Since all computations above are true for all $\varepsilon \leq \varepsilon_R$, if we integrate for $\varepsilon \in (\frac{1}{2}\varepsilon_R, \varepsilon_R)$, we get

$$\frac{\varepsilon_R}{2}\mu_t[|A^\delta x|^2_{\mathcal{W}_{\alpha_0}} \geq M]$$

$$\leq \frac{\varepsilon_R}{t}\int_{\frac{\varepsilon_R}{2}}^{\varepsilon_R}\sum_{k=0}^{n_\varepsilon} P_{x_0}[|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R]\,d\varepsilon + \frac{C\log 2}{M^\gamma}(1+|x_0|_H^2) + \frac{c_1\varepsilon_R}{2}\mathrm{e}^{-c_2\frac{R^2}{\varepsilon_R}}.$$

We use the energy inequality and the previous lemma to estimate the only complicated term in the inequality above,

$$\begin{aligned}
\frac{\varepsilon_R}{t}\int_{\frac{\varepsilon_R}{2}}^{\varepsilon_R}\sum_{k=1}^{n_\varepsilon} P_{x_0}[|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R]\,d\varepsilon &\leq \frac{\varepsilon_R}{t}\sum_{k=1}^{n_R}\int_{\frac{\varepsilon_R}{2}}^{\varepsilon_R} P_{x_0}[|\xi_{k\varepsilon}|^2_{\mathcal{V}_{\varepsilon_0}} > R]\,d\varepsilon\\
&\leq \frac{\varepsilon_R}{tR}\sum_{k=1}^{n_R}\mathbb{E}^{P_{x_0}}\left[\int_{\frac{\varepsilon_R}{2}}^{\varepsilon_R}|\xi_{k\varepsilon}|_V^2\,d\varepsilon\right]\\
&\leq \frac{\varepsilon_R}{tR}\sum_{k=1}^{n_R}\frac{1}{k}\mathbb{E}^{P_{x_0}}\left[\int_{k\frac{\varepsilon_R}{2}}^{k\varepsilon_R}|\xi_r|_V^2\,dr\right]\\
&\leq \frac{\varepsilon_R}{tR}\sum_{k=1}^{n_R}\frac{1}{k}c(1+|x_0|_H^2+k\varepsilon_R)\\
&\leq \frac{c\varepsilon_R}{R}\log\frac{1}{\varepsilon_R}(1+|x_0|_H^2),
\end{aligned}$$

where we have set $n_R = n_{\frac{\varepsilon_R}{2}}$ and $n_\varepsilon \leq n_R$ for all $\varepsilon \in [\frac{1}{2}\varepsilon_R, \varepsilon_R]$. Since by (A.5) the dependence of ε_R on R is like R^{-a}, for some exponent a depending on ε_0, we may choose R in such a way that for every $t \geq 1$,

$$\mu_t[|A^\delta x|^2_{\mathcal{W}_{\alpha_0}} \geq M] \leq \frac{c}{M^b}(1+|x_0|_H^2)\log M.$$

for a suitable $b > 0$. In conclusion, the statement of the lemma is proved for initial conditions $x_0 \in \mathcal{V}_{\varepsilon_0}$.

If $x_0 \in H$, since for every $s > 0$ we know that $\xi_s \in \mathcal{V}_{\varepsilon_0}$, P_{x_0}-a. s., then by the Markov property,

$$\begin{aligned}
\mathbb{E}^{P_{x_0}}\left[\int_s^t |A^\delta\xi_r|^{2\gamma}_{\mathcal{W}_{\alpha_0}}\,dr\right] &= \mathbb{E}^{P_{x_0}}\left[\mathbb{E}^{P_{\xi_s}}\left[\int_0^{t-s}|A^\delta\xi'_r|^{2\gamma}_{\mathcal{W}_{\alpha_0}}\,dr\right]\right]\\
&\leq C(t-s)\mathbb{E}^{P_{x_0}}[1+|\xi_s|_H^2]\\
&\leq Ct(1+|x_0|_H^2),
\end{aligned}$$

where we have used the previous lemma and this same lemma for initial conditions in $\mathcal{V}_{\varepsilon_0}$. Finally, as $s \downarrow 0$, the conclusion follows by the monotone convergence theorem. □

Proof of Theorem 3.1 Choose an arbitrary point $x_0 \in H$ and consider the sequence of measures $(\mu_t)_{t\geq 1}$ defined by formula (3.1). Since

$$\int |A^\delta x|^{2\gamma}_{\mathcal{W}_{\alpha_0}}\mu_t(dx) = \frac{1}{t}\int_0^t |A^\delta\xi_s|^{2\gamma}_{\mathcal{W}_{\alpha_0}}\,ds,$$

where the constants δ and γ are those provided by the previous lemma, it follows by that same lemma that the sequence of measures is tight in $\mathcal{W}_{\alpha_0}$. □

3.2 The Proof of Theorem 3.3

As stated in the statement of the theorem, in this section we work with the *enhanced martingale solutions* defined in Remark 2.5. It means that the energy balance [M3 − as] is available for proofs. Prior to the proof, we give a few auxiliary results, summarized in the following lemmas. In the first one we show that any solution enters in a ball of small energy with positive probability.

Lemma 3.8 (Entrance time in a ball of small energy) *Assume* [A3]. *Given* $R > 0$ *and* $\delta > 0$, *there exists* $T_1 = T_1(\delta, R)$ *such that*

$$\inf_{|x|_H^2 \le R} P(T_1, x, \{y : |y|_H^2 \le \delta\}) > 0.$$

Proof Consider a value $k_1 = k_1(\delta)$, to be chosen later, and let $A = \{\omega : \sup_{t\in[0,T_1]} |z_t|_H^2 \le k_1\}$. We know that for every x, the value $P_x[A] > 0$ is constant with respect to x. Since $|\xi_t|_H \le |z_t|_H + |v_t|_H$, we shall estimate v.

For all $\omega \in A$ such that the inequality in [M3 − as] (at page 420) holds, we have

$$\begin{aligned} |v_t|_H^2 - |v_s|_H^2 + 2\nu \int_s^t |v_r|_V^2\,dr &\le c \int_s^t |z_r|_V |\xi_r|_V |A^{\frac14} v_r|_H\,dr \\ &\le c \int_s^t (|z_r|_V |v_r|_H^{\frac12} |v_r|_V^{\frac32} + |z_r|_V^2 |v_r|_H^{\frac12} |v_r|_V^{\frac32})\,dr \\ &\le \nu \int_s^t |v_r|_V^2\,dr + k_2 \int_s^t |v_r|_H^2\,dr + k_3(t-s), \qquad (3.3) \end{aligned}$$

where we have set $k_2 = c(k_1^4 + k_1^{\frac83})$ and $k_3 = ck_1^{\frac83}$. By the Poincaré inequality (the first eigenvalue of the Laplace operator on the torus $\mathbb{T}_3$ is 1),

$$|v_t|_H^2 + (\nu - k_2) \int_s^t |v_r|_H^2\,dr \le |v_s|_H^2 + k_3(t-s),$$

and Gronwall's lemma ensures that

$$|v_t|_H^2 \le |x|_H^2 \mathrm{e}^{-(\nu-k_2)t} + \frac{k_3}{\nu - k_2}(1 - \mathrm{e}^{-(\nu-k_2)t}) \le R\mathrm{e}^{-(\nu-k_2)t} + \frac{k_3}{\nu-k_2}.$$

If we choose k_1 and T_1 in such a way that

$$k_1 \le \frac14\delta, \qquad k_2 < \nu, \qquad \frac{k_3}{\nu-k_2} \le \frac18\delta, \qquad R\mathrm{e}^{-(\nu-k_2)T_1} \le \frac18\delta,$$

we finally obtain that, if $|x|_H^2 \le R$, then $P_x[\{|\xi_{T_1}|_H^2 \le \delta\} \cap A] = P_x[A]$. □

The second lemma shows that with positive probability the dynamics enters into a (sufficiently large) ball of space V.

Lemma 3.9 (Entrance time in a ball of finite dissipation) *Assume* [A3] *from Assumption* 1.1. *Then there exists* $\delta > 0$ *small enough such that there are* $T_2 = T_2(\delta) > 0$ *and* $R_2 = R_2(\delta) > 0$

with

$$\inf_{|x|_H^2 \le \delta} P(T_2, x, \{y : |y|_V^2 \le R_2\}) > 0.$$

Proof Set $T_2 = 1$ and let $A = \{\sup_{[0,1]} |A^{\frac{5}{8}} z|_H^2 \le k_1\}$, with k_1 to be chosen later, together with δ.

For all $\omega \in A$ for which the inequality in [M3-as] (at p. 420) holds, we can proceed as in the proof of Lemma 3.8 to get $|v_t|_H^2 \le \delta + \frac{k_3}{\nu - k_2}$, with k_2 and k_3 defined similarly. Using (3.3), we get

$$\int_0^1 |v_s|_V^2\, ds \le \frac{1}{\nu}\left[\delta + k_3 + k_2\left(\delta + \frac{k_3}{\nu - k_2}\right)\right] := k_4,$$

where k_1 is small enough so that $k_2 < \nu$.

Next, we notice that the set $\{r \in [0,1] : |v_r|_V^2 \le 2k_4\}$ is non-empty (its Lebesgue measure is larger than one half). So for each r_0 in such a set, $|v_{r_0}|_V^2 \le 2k_4$. Since the energy inequality $[\mathrm{M3}-\mathrm{as}]$ holds, for a short time after r_0, v coincides with the unique regular solution. We shall choose k_1 and δ small enough so that the short time goes well beyond 1.

Indeed, using (2.1) (as in (A.6) with $\varepsilon_0 = \frac{1}{4}$), we get for suitable universal constants c_1 and c_2,

$$\frac{d}{dt}|v|_V^2 + 2\nu|Av|_H^2 \le \nu|Av|_H^2 + c_1(|v|_V^6 + |A^{\frac{5}{8}} z|_H^4) \le \nu|Av|_H^2 + c_1(|v|_V^6 + c_2 k_1^4),$$

and so, if $\varphi(r) = |v_r|_V^2 + k_1^{\frac{4}{3}}$, we have $\varphi(r_0) \le 2k_4 + k_1^{\frac{4}{3}}$ and $\dot\varphi \le c_1\varphi^3$. Now, if we choose k_1 and δ small enough so that

$$4c_1(2k_4 + k_1^{\frac{4}{3}})^2 \le 1$$

the solution to the differential inequality of φ is finite at least up to time $1 + r_0$. In particular, $\varphi(1) \le (2c_1)^{-\frac{1}{2}}$ and so by easy computations,

$$|\xi_{T_2}|_V^2 = |\xi_1|_V^2 \le (|v_1|_V + |z_1|_V)^2 \le 2k_1 + \frac{2}{\sqrt{2c_1}}.$$

We choose now the last term on the right-hand side of the above formula as R_2. In conclusion, $P(T_2, x, \{|y|_V^2 \le R_2) \ge P_x[A]$ and again the value of $P_x[A]$ is independent of x. □

In the last auxiliary lemma we show that the dynamics enters in a compact subset of $\mathcal{W}_{\alpha_0}$. This is crucial since the strong Feller property holds in the topology of $\mathcal{W}_{\alpha_0}$ (Theorem 2.7).

Lemma 3.10 (Entrance time in a ball of high regularity) *Assume* [A3] *from Assumption 1.1. Then there is $\beta > 0$ (depending only on α_0) such that for every $R_2 > 0$ there are a time $T_3 = T_3(R_2) > 0$ and a number $C = C(R_2) > 0$ and*

$$\inf_{|x|_V^2 \le R_2} P(T_3, x, \{y : |A^\beta y|_{\mathcal{W}_{\alpha_0}}^2 \le C\}) > 0.$$

Proof Given $R_2 > 0$, we choose $\beta = \theta'' - \theta(\alpha_0)$, T_3 and C as given in Lemma A.3. Notice that the set $K = \{y : |A^\beta y|_{\mathcal{W}_{\alpha_0}}^2 \le C\}$ is a compact subset of $\mathcal{W}_{\alpha_0}$.

If $\tau = \tau^{(\frac{1}{4},3R_2)}$ is the time up to which all solutions starting at x coincide with the unique solution to problem (A.1), then

$$\begin{aligned}P(T_3, x, K) &= P_x[|A^\beta \xi_{T_3}|^2_{\mathcal{W}_{\alpha_0}} \le C]\\ &\ge P_x[|A^\beta \xi_{T_3}|^2_{\mathcal{W}_{\alpha_0}} \le C,\ \tau > T_3]\\ &\ge P_x^{(\frac{1}{4},3R_2)}[|A^\beta \xi_{T_3}|^2_{\mathcal{W}_{\alpha_0}} \le C,\ \tau > T_3].\end{aligned}$$

Now, the conclusion follows from Lemma A.3. □

Proof of Theorem 3.3 Let $(P_x)_{x\in H}$ be a Markov solution and consider the corresponding transition kernel $(P(t,x,\cdot))_{t\ge 0, x\in H}$. We choose the value of ε_0 given in Lemma A.3 and we consider the value $\theta'' > \theta(\alpha_0)$ provided by the same lemma.

The exponential convergence follows from an abstract result of Goldys & Maslowski [15, Theorem 3.1] (which, in turns, is based on results from the book by Meyn & Tweedie [16]). More precisely, we need to verify the following four conditions,

1. the measures $(P(t,x,\cdot))_{t>0,x\in H}$ are equivalent,
2. $x \to P(t,x,\Gamma)$ is continuous in $\mathcal{W}_{\alpha_0}$ for all $t > 0$ and Borel sets $\Gamma \subset H$,
3. For each $R \ge 1$ there are $T_0 > 0$ and a compact subset $K \subset \mathcal{W}_{\alpha_0}$ such that

$$\inf_{|x|^2_H \le R} P(T_0, x, K) > 0,$$

4. there are $k, b, c > 0$ such that for all $t \ge 0$,

$$\mathbb{E}^{P_x}[|\xi_t|^2_H] \le k|x|^2_H \mathrm{e}^{-bt} + c.$$

The first property follows from Theorem 13 in [12] (there equivalence is stated only for $x \in \mathcal{W}_{\alpha_0}$, but it easy to see by the Markov property that it holds for $x \in H$, as $\mathcal{W}_{\alpha_0}$ is a set of full measure for each $P(t,x,\cdot)$). The second property follows from the strong Feller property, while the fourth property follows from Lemma 3.6.

We only need to prove the third property. We fix $R \ge 1$ and we wish to prove that there are $T_0 = T_0(R)$ and $K = K(R)$ such that

$$\inf_{|x|^2_H \le R} P(T_0, x, \{y : |A^{\theta''} y|^2_H \le K\}) > 0. \tag{3.4}$$

We choose the value δ provided by Lemma 3.9 together with the time T_2 and value R_2. Corresponding to the values R and δ, Lemma 3.8 gives a time T_1. Moreover, corresponding to R_2, Lemma 3.10 provides the time T_3 and value C.

We set $T_0 = T_1 + T_2 + T_3$, then if $|x|^2_H \le R$, using three times the Markov property,

$$\begin{aligned}&P(T_0, x, K)\\ &\quad\ge \inf_{|x_3|^2_V \le R_2} P(T_3, x_3, K) \inf_{|x_2|^2_H \le \delta} P(T_2, x_2, \{|z|^2_V \le R_2\}) \inf_{|x_1|^2_H \le R} P(T_1, x_1, \{|y|^2_H \le \delta\})\end{aligned}$$

and the right-hand side is positive (and bounded from below independently of x) due to Lemma 3.8, 3.9 and 3.10.

Finally, the constants C_{exp} and a in the statement of the theorem are independent of the Markov solution since all computations either depend on the data (the viscosity ν, the strength of the noise σ^2, etc., such as in Lemma 3.6) or are made on the regularized problem analysed in the appendix. $\square$

4 Further Analysis of Equilibrium States

In the previous section we have shown that, under suitable assumptions on the driving noise, every Markov solution has a unique invariant measure. As in principle there can be several different Markov solutions, so are invariant measures.

In the first part of the section we show that well-posedness of the martingale problem associated to (1.2) is equivalent to the statement that there is only one invariant measure, regardless of the multiplicity of solutions.

In the second part we give some remarks on symmetries of invariant measures, while in the third part we analyse the energy balance.

4.1 A Connection Between Uniqueness of Invariant Measures and Well-Posedness of the Martingale Problem

4.1.1 Stationary Solutions

Consider the (unique) invariant measure associated to a Markov solution $(P_x)_{x\in H}$, as provided by Corollary 3.2, and define the following probability measure

$$P_\star = \int P_x\, \mu_\star(dx). \tag{4.1}$$

Lemma 4.1 *The probability measure $P_\star$ defined above is invariant* (*in the following*, stationary) *with respect to the time shifts $\eta_t : \Omega \to \Omega$ defined as*

$$\eta_t(\omega)(s) = \omega(t+s).$$

Proof It is sufficient to prove that the finite dimensional marginals of $P_\star$ and $\eta_s P_\star$ are the same. The case of one single time is easy, by invariance of $\mu_\star$. We consider only the two-dimensional case (one can proceed by induction in the general case). Consider $t_1 < t_2$, then by the Markov property and invariance of $\mu_\star$,

$$\begin{aligned}
\mathbb{E}^{\eta_s P_\star}[f(\xi_{t_1}, \xi_{t_2})] &= \int \mathbb{E}^{P_x}[f(\xi_{s+t_1}, \xi_{s+t_2})]\, \mu_\star(dx) \\
&= \int \mathbb{E}^{P_x}\Big[\mathbb{E}^{P_{\omega(s+t_1)}}[f(\omega(s+t_1), \xi_{t_2-t_1})]\Big]\, \mu_\star(dx) \\
&= \int \mathbb{E}^{P_x}[F(\xi_{s+t_1})]\, \mu_\star(dx) = \int \mathbb{E}^{P_x}[F(\xi_{t_1})]\, \mu_\star(dx) \\
&= \int \mathbb{E}^{P_x}[f(\xi_{t_1}, \xi_{t_2})]\, \mu_\star(dx) = \mathbb{E}^{P_\star}[f(\xi_{t_1}, \xi_{t_2})],
\end{aligned}$$

where in the above formula we have set $F(y) = \mathbb{E}^{P_y}[f(y, \xi_{t_2-t_1})]$. $\square$

In turns, the lemma above ensures that $P_\star$ is the unique probability measure on Ω such that

1. $P_\star$ is stationary,
2. $P_\star$ is associated[2] to the Markov solution $(P_x)_{x\in H}$.

Uniqueness follows easily since $\mu_\star$ is the unique invariant measure of the Markov solution $(P_x)_{x\in H}$ and since the law of a Markov process is determined by its one-dimensional (with respect to time) marginal distributions (as in the proof of the lemma above). We shall see later on that for a special class of invariant measures this uniqueness statement can be strengthened (see Proposition 4.4).

In general one can have several stationary solutions (see for example [19] for the definition and a different proof of existence) and possibly not all of them are associated to a Markov solution. Hence we define the two sets,

$$\mathcal{I} = \{\mu \in \Pr(H) : \mu \text{ is the marginal of a stationary solution}\} \tag{4.2}$$

$$\mathcal{I}_m = \left\{\mu \in \Pr(H) : \begin{array}{l}\mu \text{ is the unique invariant measure associated to a}\\ \text{Markov solution}\end{array}\right\} \tag{4.3}$$

and, trivially, $\mathcal{I}_m \subset \mathcal{I}$.

Remark 4.2 (Topological properties of $\mathcal{I}$ and $\mathcal{I}_m$) By the same properties that ensure existence of solutions (and following similar computations, see for example [11]), it is easy to see that $\mathcal{I}$ is a compact subset of Ω. Moreover, by Corollary 3.2, $\mathcal{I}_m$ and hence $\mathcal{I}_e$ are relatively compact in a much stronger topology, where $\mathcal{I}_e$ is defined later in (4.4).

4.1.2 A Short Recap on the Selection Principle

It is necessary to give a short account on the procedure which proves the existence of Markov selection (namely, the proof of Theorem 2.6). We refer to [11] for all details.

Given $x \in H$, let $\mathcal{C}(x) \subset \Pr(\Omega)$ be the set of all weak martingale solutions (according to Definition 2.4) to equation (1.2), starting at x.

In the proof of Theorem 2.6 (see [11]) the sets $\mathcal{C}(x)$ are shrunken to one single element in the following way. Fix a family $(\lambda_n, f_n)_{n\geq 1}$ which is dense in $[0,\infty) \times C_b(D(A)')$ and consider the functionals $J_n = J_{\lambda_n, f_n}$, where $J_{\lambda,f}$ is given by

$$J_{\lambda,f}(P) = \mathbb{E}^P\left[\int_0^\infty \mathrm{e}^{-\lambda t} f(\xi_t)\,dt\right]$$

for arbitrary $\lambda > 0$ and $f : D(A)' \to \mathbf{R}$ upper semi-continuous. Next, set

$$\mathcal{C}_0(x) = \mathcal{C}(x), \qquad \mathcal{C}_n(x) = \left\{P \in \mathcal{C}_{n-1}(x) : J_n(P) = \sup_{Q\in\mathcal{C}_{n-1}(x)} J_n(Q)\right\}.$$

All these sets are compact and their intersection is a single element (the selection associated to this maximized sequence), $\bigcap_{n\in\mathbf{N}} \mathcal{C}_n(x) = \{P_x\}$.

Given now a probability measure μ on H, one can define the set $\mathcal{C}(\mu)$ as the set of all probability measures P on Ω such that

1. the marginal at time 0 of P is μ;

[2]We say that a probability measure P on Ω is associated to a Markov solution $(P_x)_{x\in H}$ if for every $t \geq 0$, $P|^{\omega}_{\mathcal{B}_t} = P_{\omega(t)}$ for P-a. e. $\omega \in \Omega$, where $(P|^{\omega}_{\mathcal{B}_t})_{\omega\in\Omega}$ is a regular conditional probability distribution of P given $\mathcal{B}_t$.

2. there is a map $x \mapsto Q_x : H \to \Pr(\Omega)$ such that $P = \int Q_x\, \mu(dx)$ and $Q_x \in \mathcal{C}(x)$ for all x (in different words, the conditional distribution of P at time 0 is made of elements from sets $(\mathcal{C}(x))_{x\in H}$).

We can now give the following extension to the selection principle.

Proposition 4.3 *Let $(P_x)_{x\in H}$ be the Markov selection associated to the sequence $(\lambda_n, f_n)_{n\geq 1}$. Then the probability $P_\mu = \int_H P_x\, \mu(dx)$ is the unique maximizer associated to the sequence $(\lambda_n, f_n)_{n\geq 1}$. More precisely,*

$$\begin{aligned}
J_1(P_\mu) &= \sup_{P\in\mathcal{C}_0(\mu)} J_1(P),\\
\dots &\qquad \dots,\\
J_n(P_\mu) &= \sup_{P\in\mathcal{C}_{n-1}(\mu)} J_n(P),\\
\dots &\qquad \dots.
\end{aligned}$$

Proof Since each $Q \in \mathcal{C}(\mu)$ is given by $Q = \int Q_x\, \mu(dx)$, for some $x \mapsto Q_x$, by linearity of the map J_1 it easily follows that $P_\mu \in \mathcal{C}_1(\mu)$. Moreover, each $Q \in \mathcal{C}_1(\mu)$ has a similar structure: $Q = \int Q_x\, \mu(dx)$ and $Q_x \in \mathcal{C}_1(x)$ for μ-a. e. $x \in H$. In fact, $J_1(Q_x) \leq J_1(P_x)$, μ-a. s., and $J_1(Q) = J_1(P_\mu)$, and so $J_1(Q_x) = J_1(P_x)$, for μ-a. e. x. By induction, $P_\mu \in \mathcal{C}_n(\mu)$ and for each $Q = \int Q_x\, \mu(dx) \in \mathcal{C}_n(\mu)$, $Q_x \in \mathcal{C}_n(x)$, for μ-a. e. $x \in H$.

In conclusion, $P_\mu \in \mathcal{C}_\infty(\mu) = \bigcap \mathcal{C}_n(\mu)$ and for each $Q = \int Q_x\, \mu(dx) \in \mathcal{C}_\infty(\mu)$, $Q_x \in \mathcal{C}_\infty(x)$, for μ-a. e. $x \in H$. But we know that each $\mathcal{C}_\infty(x)$ has exactly one element, P_x, so that in conclusion the only element of $\mathcal{C}_\infty(\mu)$ is P_μ. □

4.1.3 Connection with Well-Posedness

Now, if we are given a sequence $(\lambda_n, f_n)_{n\in\mathbf{N}}$ as above, the selection principle provides a Markov solution $(P_x)_{x\in H}$. Corollary 3.2 ensures that this Markov solution has a unique invariant measure $\mu_\star$. Moreover, from the proposition above, the stationary solution $P_\star = \int P_x \mu_\star(dx)$ is the unique sequential maximizer of the sequence $(J_n)_{n\in\mathbf{N}}$ on $\mathcal{C}(\mu_\star)$. This justifies, in analogy with the definition of (4.2) and (4.3), the definition of the following set,

$$\mathcal{I}_e = \left\{ \mu : \begin{array}{l} \mu \text{ is the invariant measure associated to a Markov solution} \\ \text{obtained by the maximization procedure for some sequence} \\ (\lambda_n, f_n)_{n\in\mathbf{N}} \end{array} \right\}, \tag{4.4}$$

and obviously $\mathcal{I}_e \subset \mathcal{I}_m \subset \mathcal{I}$.

Proposition 4.4 *If $\mu_\star \in \mathcal{I}_e$, then the stationary solution $P_\star$ associated to $\mu_\star$ is the unique stationary measure in $\mathcal{C}(\mu_\star)$.*

Proof Since $\mu_\star \in \mathcal{I}_e$, by definition there is a sequence $(\lambda_n, f_n)_{n\in\mathbf{N}}$ dense in $[0,\infty) \times C_b(D(A)')$ such that $P_\star$ maximizes functionals $J_n = J_{\lambda_n, f_n}$ (one after the other, as explained in Proposition 4.3). Now, if $\tilde{P} \in \mathcal{C}(\mu_\star)$ is a stationary solution, then

$$\begin{aligned}
J_n(\tilde{P}) &= \mathbb{E}^{\tilde{P}}\left[\int_0^\infty \mathrm{e}^{-\lambda_n t} f_n(\xi_t)\, dt\right] = \left(\int f_n(x)\, \mu_\star(dx)\right) \int_0^\infty \mathrm{e}^{-\lambda_n t}\, dt\\
&= \frac{1}{\lambda_n} \int f_n(x)\, \mu_\star(dx),
\end{aligned}$$

and so $J_n(\tilde{P}) = J_n(P_\star)$ for all n. By Proposition 4.3, it follows that $\tilde{P} = P_\star$. □

If we consider now Markov solutions sharing the same properties as those obtained for the Navier-Stokes equations, namely each of them is strong Feller and irreducible on $\mathcal{W}_{\alpha_0}$, the previous result provides immediately a criterion for well-posedness. In few words, uniqueness of the invariant measures among Markov solutions is equivalent to well-posedness of the martingale problem.

Corollary 4.5 *Assume that every Markov selection is $\mathcal{W}_{\alpha_0}$–strong Feller and fully supported on $\mathcal{W}_{\alpha_0}$. If $(P_x)_{x\in H}$ and $(P'_x)_{x\in H}$ are two Markov selections, with $(P_x)_{x\in H}$ coming from a maximization procedure, and they have the same invariant measure, then they coincide on $\mathcal{W}_{\alpha_0}$.*

Proof Let $P_\star$ and $P'_\star$ be the stationary solutions associated to the two selections. If the two selections have the same invariant measure, it follows from the previous theorem that they have the same stationary solution, that is $P_\star = P'_\star$. It follows from this that their conditional probability distributions at time 0 coincide,

$$P_x = P'_x, \qquad \mu_\star\text{-almost surely.}$$

By $\mathcal{W}_{\alpha_0}$-strong Feller regularity and irreducibility they coincide on every $x \in \mathcal{W}_{\alpha_0}$. □

We summarize the result in the following theorem. It follows easily from the previous corollary and from the fact that well-posedness of the martingale problem is equivalent to uniqueness of Markov selections (see Theorem 12.2.4 of Stroock & Varadhan [20]).

Theorem 4.6 *Under* [A4] *of Assumption 1.1, assume that the set $\mathcal{I}_e$ defined in (4.4) contains only one invariant measure. Then the martingale problem associated to the Navier-Stokes equations (1.2) is well-posed on $\mathcal{W}_{\alpha_0}$ (and hence on $\mathcal{V}_{\varepsilon_0}$ for all $\varepsilon_0 > 0$).*

Proof We only have to prove that, given two Markov solutions $(P_x)_{x\in H}$ and $(P'_x)_{x\in H}$, for every $x \in \mathcal{V}_{\varepsilon_0}$ we have $P_x = P'_x$. This statement holds for $x \in \mathcal{W}_{\alpha_0}$ by the previous corollary. Fix $\varepsilon_0 > 0$ and $x \in \mathcal{V}_{\varepsilon_0}$. Choose $R \gg |x|^2_{\mathcal{V}_{\varepsilon_0}}$, then, for every bounded continuous ϕ, by the Markov property,

$$\mathcal{P}_t\phi(x) = \mathbb{E}^{P_x}[\mathcal{P}_{t-\delta}\phi(\xi_\delta)\mathbb{1}_{\{\tau^{(\varepsilon_0,R)}>\delta\}}] + \mathbb{E}^{P_x}[\mathcal{P}_{t-\delta}\phi(\xi_\delta)\mathbb{1}_{\{\tau^{(\varepsilon_0,R)}\leq\delta\}}],$$

where $\mathcal{P}$ is the transition semigroup of $(P_x)_{x\in H}$. The first term on the right-hand side is independent of the selection, by the weak-strong uniqueness of Theorem A.1, hence

$$\mathcal{P}_t\phi(x) - \mathcal{P}'_t\phi(x) = \mathbb{E}^{P_x}[\mathcal{P}_{t-\delta}\phi(\xi_\delta)\mathbb{1}_{\{\tau^{(\varepsilon_0,R)}\leq\delta\}}] - \mathbb{E}^{P'_x}[\mathcal{P}'_{t-\delta}\phi(\xi_\delta)\mathbb{1}_{\{\tau^{(\varepsilon_0,R)}\leq\delta\}}],$$

where $\mathcal{P}'$ is the transition semigroup of $(P'_x)_{x\in H}$. By the blow-up estimate of Theorem A.1, as $\delta \to 0$, we get $\mathcal{P}_t\phi(x) = \mathcal{P}'_t\phi(x)$ for all ϕ and all t. □

4.2 Translations-Invariance and Other Symmetries

In the analysis of homogeneous isotropic turbulence, for which (1.1) can be considered a model, it is interesting to consider equilibrium states invariant with respect to several symmetries (see for example Frisch [14]).

Here we are interested in solutions which are *translations-invariant* (in the physical space). For every $a \in \mathbf{R}^3$, define on $\mathcal{D}^\infty$ the map $m_a : \mathcal{D}^\infty \to \mathcal{D}^\infty$ as

$$m_a(\varphi)(x) = \varphi(a + x), \quad x \in \mathbf{R}^3$$

for any $\varphi \in \mathcal{D}^\infty$. The map obviously extends to H and $D(A^\alpha)$ for each α. By composition, it extends to continuous functions on H (or $D(A^\alpha)$ for every α) and, by duality, to probability measures on H. It also extends to Ω as

$$m_a(\omega)(t) = m_a(\omega(t)), \quad t \geq 0, \ \omega \in \Omega,$$

and, by duality, to probability measures on Ω.

A function (or a measure) is *translations-invariant* if it is invariant under the action of $(m_a)_{a \in \mathbf{R}^3}$. The Navier-Stokes equations are translations-invariant, so equation (1.2) is translations-invariant only if such is the noise. The driving noise is translations-invariant if and only if the covariance $\mathcal{Q}$ commutes with all m_a. It is easy to verify that this is equivalent to have homogeneous noise which, in turns, is equivalent to have that $\mathcal{Q}$ is diagonal on the Fourier basis. So, easy examples of homogeneous noise compatible with the properties of Assumption 1.1 are $\mathcal{Q} = A^{-\frac{3}{2}-\alpha_0}$ for any α_0 in the correct range.

Proposition 4.7 *Assume that $\mathcal{Q}$ is diagonal on the Fourier basis. Then the following properties hold true.*

1. *For every $a \in \mathbf{R}^3$, m_a is a one-to-one map on $\mathcal{I}$, on $\mathcal{I}_m$ and on $\mathcal{I}_e$.*
2. *There is at least one translations-invariant measure in $\mathcal{I}$.*

Proof We first show that if P is the law of a solution to equations (1.2), then $m_a P$ is also a solution for every $a \in \mathbf{R}^3$. Since for every $a \in \mathbf{R}^3$ the map m_a is an isometry on H, the image of a cylindrical Wiener process on H is again a cylindrical Wiener process. The assumption on $\mathcal{Q}$ ensures now that the noise term is translations-invariant and so it is easy to check that all requirements of either Definition 2.4 or of any definition of solutions for the stochastic PDE (1.1) available in the literature (see for example Flandoli & Gątarek [8], we refer also to [19] as regarding stationary solutions) are verified.

In particular, if P is stationary, then $m_a P$ is again stationary and so m_a is a one-to-one map on $\mathcal{I}$. Moreover, since $\mathcal{I}$ is closed and convex (see Remark 4.2), it follows that there exists a translations-invariant measure. Indeed, given $\mu \in \mathcal{I}$, there is a stationary solution P_μ whose marginal is μ. Now, the probability measure

$$\tilde{P} = \frac{1}{|\mathbb{T}_3|} \int_{\mathbb{T}_3} m_a P_\mu \, da$$

is again a stationary solution and its marginal is translations-invariant, as $m_{a+2\pi k} = m_a$ for every $k \in \mathbf{Z}^3$.

We next prove that m_a maps $\mathcal{I}_m$ one-to-one. Let $\mu_\star \in \mathcal{I}_m$ and consider a Markov solution $(P_x)_{x \in H}$ having $\mu_\star$ as one of its invariant measures. Fix $a \in \mathbf{R}^3$ and set $Q_x = m_a(P_{m_{-a}(x)})$. It is easy to verify that $(Q_x)_{x \in H}$ is another Markov solution, since

$$Q_x|^\omega_{\mathcal{B}_t} = m_a(P_{m_{-a}(x)})|^\omega_{\mathcal{B}_t} = m_a(P_{m_{-a}(\omega)(t)}), \quad P_{m_{-a}(x)}\text{-a. s.}$$

Moreover, $m_a(\mu)$ is an invariant measure of $(Q_x)_{x \in H}$.

Finally, in order to show that m_a maps $\mathcal{I}_e$ one-to-one, we only need to find a maximizing sequence for the solution $(Q_x)_{x\in H}$ defined above. Let $(\lambda_n, f_n)_{n\in\mathbf{N}}$ be a maximizing sequence for $(P_x)_{x\in H}$, then $(\lambda_n, f_n \circ m_{-a})_{n\in\mathbf{N}}$ is a maximizing sequence for $(Q_x)_{x\in H}$. $\square$

We stress that in the proposition above existence of a translations invariant equilibrium measure is granted in $\mathcal{I}$, but we do not know if such a measure belongs to $\mathcal{I}_m$.

Notice finally that if problem (1.2) is well-posed, it follows easily that the unique invariant measure must be *translations-invariant*.

Similar conclusions can be found for other symmetries of the torus, such as isotropy (invariance with respect to rotations, see for example [9] where such symmetries are discussed in view of a connections between homogeneous turbulence and (1.1)).

4.3 The Balance of Energy

In the framework of Markov solutions examined in this paper, the balance of energy corresponds to the a. s. super-martingale property [M3] (and, more generally, of [M4]) of Definition 2.4. As clarified in [11], the two facts

1. the balance holds only for almost every time,
2. the balance is an inequality, rather than an equality,

correspond to a lack of regularity, in time in the first case and in space in the second, of solutions to the equations (1.1). From the point of view of the model, such facts translate to a loss of energy in the balance.

Generally speaking, the problem could be approached by using the Doob-Meyer decomposition (which may hold even in this case, where the energy-balance process $\mathcal{E}^1$ is not continuous and the filtration $(\mathcal{B}_t)_{t\geq 0}$ does not satisfy the usual conditions, see Dellacherie & Meyer [6]). We shall follow a different approach, due to the lack of regularity of trajectories solutions to the equations. We shall see that the *bounded variation* term in the decomposition of $\mathcal{E}^1$ is a distribution valued process.

Let $a > 0$ and define the operator $L_a = \exp(-aA^{\frac{1}{2}})$. Given a martingale solution P_x starting at some $x \in H$, there is a Wiener process $(W_t)_{t\geq 0}$ such that the canonical process ξ on Ω solves (1.2). The process $L_a\xi$ under P_x is regular enough so that we can use the standard stochastic calculus. Given an arbitrary $\varphi \in C_c^\infty(\mathbf{R})$, with support in $[0,\infty)$, Itô formula gives

$$\begin{aligned} d[\varphi(t)|L_a\xi_t|_H^2] &= \varphi'(t)|L_a\xi_t|_H^2 + 2\varphi(t)\langle L_a\xi_t, dL_a\xi_t\rangle_H + \varphi(t)\sigma_a^2 \\ &= \varphi'(t)|L_a\xi_t|_H^2 - 2\nu\varphi(t)|L_a\xi_t|_V^2 - 2\varphi(t)\langle L_a\xi_t, L_aB(\xi_t,\xi_t)\rangle_H \\ &\quad + 2\varphi(t)\langle L_{2a}\xi_t, \mathcal{Q}^{\frac{1}{2}}\,dW\rangle_H + \varphi(t)\sigma_a^2, \end{aligned}$$

where $\sigma_a^2 = \mathrm{Tr}[\mathcal{Q}L_{2a}]$, and so, by integrating in time,

$$\begin{aligned} &2\nu\int \varphi(t)|L_a\xi_t|_V^2\,dt + 2\int \varphi(t)\langle L_a\xi_t, L_aB(\xi_t,\xi_t)\rangle_H\,dt \\ &\quad = \int \varphi'(t)|L_a\xi_t|_H^2\,dt + 2\int \varphi(t)\langle L_{2a}\xi_t, \mathcal{Q}^{\frac{1}{2}}\,dW\rangle_H\,dt + \sigma_a^2\int \varphi(t)\,dt, \end{aligned}$$

P_x-a. s. As $a \downarrow 0$, the operator L_a approximates the identity, so that by the regularity of ξ under P_x,

$$
\begin{aligned}
&2\nu \int \varphi(t)|L_a\xi_t|_V^2\,dt \longrightarrow 2\nu \int \varphi(t)|\xi_t|_V^2\,dt,\\
&\int \varphi'(t)|L_a\xi_t|_H^2\,dt \longrightarrow \int \varphi'(t)|\xi_t|_H^2\,dt,\\
&2\int \varphi(t)\langle L_{2a}\xi_t, \mathcal{Q}^{\frac12}\,dW\rangle_H\,dt \longrightarrow 2\int \varphi(t)\langle \xi_t, \mathcal{Q}^{\frac12}\,dW\rangle_H\,dt,\\
&\sigma_a^2\int \varphi(t)\,dt \longrightarrow \sigma^2\int \varphi(t)\,dt,
\end{aligned}
$$

P_x-a. s. and in $L^1(\Omega)$, where $\sigma^2 = \mathrm{Tr}[\mathcal{Q}]$. In conclusion, the limit

$$
\int \mathcal{J}_t(\xi)\varphi(t)\,dt := \lim_{a\downarrow 0}\int_0^\infty \varphi(t)\langle L_a\xi_t, L_a B(\xi_t,\xi_t)\rangle\,dt \tag{4.5}
$$

exists P_x-a. s. and in $L^1(\Omega)$, and defines a distributions-valued random variable. Moreover, $\mathcal{J}(\xi)$ depends only on ξ (that is, on P_x) and not on the approximation operators $(L_a)_{a>0}$ used. We finally have

$$
\begin{aligned}
&2\nu\int \varphi(t)|\xi_t|_V^2\,dt + 2\int \varphi(t)\mathcal{J}_t(\xi)\,dt\\
&\quad = \int \varphi'(t)|\xi_t|_H^2\,dt + 2\int \varphi(t)\langle \xi_t, \mathcal{Q}^{\frac12}\,dW\rangle_H\,dt + \sigma^2\int \varphi(t)\,dt.
\end{aligned}
\tag{4.6}
$$

The previous computations and Lemma 2.3 provide finally the following result. In few words, the next theorem states that the term $\mathcal{J}(\xi)$ plays the role of the increasing process in the Doob-Meyer decomposition of the a. s. super-martingale $(\mathcal{E}_t^1)_{t\ge 0}$ (defined by property [M3] of Definition 2.4).

Theorem 4.8 *Given a martingale solution P_x, there exists a distribution-valued random variable $\mathcal{J}(\xi)$, defined by (4.5), such that the (distribution-valued) process $\mathcal{E}_t^1 + 2\mathbb{E}^{P_x}[\int \mathcal{J}_r(\xi)\,dr|\mathcal{B}_t]$ is a distribution-valued martingale, that is for every $s\ge 0$ and every $\varphi\in C_c^\infty(\mathbf{R})$ with* $\mathrm{Supp}\,\varphi\subset[s,\infty)$,

$$
\mathbb{E}^{P_x}\left[\int \varphi'(t)\left(\mathcal{E}_t^1 + 2\mathbb{E}^{P_x}\left[\int \mathcal{J}_r(\xi)\,dr\middle|\mathcal{B}_t\right]\right)dt\,\middle|\,\mathcal{B}_s\right] = 0.
$$

Moreover, $\mathcal{J}(\xi)$ is a positive distribution, in the sense that for every $\varphi\in C_c^\infty(\mathbf{R})$ with $\varphi\ge 0$ and $\mathrm{Supp}\,\varphi\subset[s,\infty)$,

$$
\mathbb{E}^{P_x}\left[\int \varphi(r)\mathcal{J}_r(\xi)\,dr\,\middle|\,\mathcal{B}_s\right] \ge 0. \tag{4.7}
$$

Proof The first part of the theorem follows easily, since

$$
\begin{aligned}
\int \varphi'(t)\mathcal{E}_t^1\,dt &= \int \varphi'(t)\left(|\xi_t|_H^2 + 2\nu\int_0^t |\xi_r|_V^2\,dr - \sigma^2 t\right)dt\\
&= \int \varphi'(t)|\xi_t|_H^2\,dt - 2\nu\int \varphi(t)|\xi_t|_V^2\,dt + \sigma^2\int \varphi(t)\,dt,
\end{aligned}
$$

and so, using the above computation and formula (4.6), we get the conclusion. The second part is a consequence of the first part (the martingale property) and the fact that $(\mathcal{E}_t^1)_{t\geq 0}$ is an a. s. super-martingale. □

Remark 4.9 The Itô formula applied to $\varphi(t)|L_a\xi|_H^{2n}$ provides an analysis of the a. s. super-martingale $\mathcal{E}^n$, defined in property [M4] of Definition 2.4, similar to that developed above for $\mathcal{E}^1$ and $\mathcal{J}(\xi)$.

Remark 4.10 Duchon & Robert [7] show that the energy equality holds for suitable weak solutions to the deterministic Navier-Stokes equations if one takes into account the additional term $\mathcal{D}$, a distribution in space and time, obtained by means of the limit of space-time regularizations. Their computations in our setting would lead to a random distribution $\mathcal{D}(\xi)(t,x)$ in space and time and

$$\mathcal{J}(\xi)(t) = \langle \mathcal{D}(\xi), \mathbb{1}_{\mathbb{T}_3}\rangle = \int_{\mathbb{T}_3} \mathcal{D}(\xi)(t,x)\,dx.$$

This is only formal because in principle our solutions are not *suitable* (see [19] for existence of suitable solutions in the stochastic setting).

Moreover, they relate the quantity $\mathcal{D}$ to the four-fifth law in turbulence theory (see for instance Frisch [14]).

4.3.1 The Mean Rate of Inertial Energy Dissipation

Consider a Markov solution $(P_x)_{x\in H}$ and let $\mu_\star$ be its unique invariant measure. Define the *mean rate of energy dissipation* as

$$\varepsilon(\mu_\star) := \mathbb{E}^{\mu_\star}[|x|_V^2].$$

We know that $2\nu\varepsilon(\mu_\star)\leq\sigma^2$. We can as well consider the expectation with respect to the stationary solution $P_{\mu_\star}$ of the distribution $\mathcal{J}$ defined in the previous section. As $\mu_\star$ is an invariant measure, the distribution $\varphi \longrightarrow \mathbb{E}^{P_{\mu_\star}}[\langle\mathcal{J}(\xi),\varphi\rangle]$ is invariant with respect to time-shifts. Hence there is a constant $\iota(\mu_\star)$, that we call *mean rate of inertial energy dissipation*, such that

$$\mathbb{E}^{P_{\mu_\star}}\left[\int \mathcal{J}_t(\xi)\varphi(t)\,dt\right] = \iota(\mu_\star)\int_0^\infty \varphi(x)\,dx.$$

We notice that, as a consequence of (4.7),

$$0\leq \iota(\mu_\star).$$

By taking the expectation in the balance of energy given in (4.6), we finally obtain the following energy equality,

$$\nu\varepsilon(\mu_\star) + \iota(\mu_\star) = \frac{1}{2}\sigma^2. \tag{4.8}$$

The quantity $\iota(\mu_\star)$ can be given as the expectation of (4.5). Notice that in this case the expectation in $\mu_\star$ and the limit in (4.5) commute.

We give a different formulation of $\iota(\mu_\star)$ in terms of Fourier modes. As the definition of $\mathcal{J}$ (and hence of ι) is independent of the approximation (as long as the approximating quantities are regular enough, so that all the computations are correct), we use a ultraviolet cut-off in the Fourier space. For every threshold K, define the projection $\mathcal{P}^l_K$ of H onto low modes as

$$\mathcal{P}^l_K x = \sum_{|k|_\infty \le N} x_k \mathrm{e}^{\mathrm{i}k\cdot x}, \quad \text{for } x = \sum_{k\in\mathbf{Z}^3} x_k \mathrm{e}^{\mathrm{i}k\cdot x},$$

and the projection onto high modes $\mathcal{P}^h_K = I - \mathcal{P}^l_K$. Applying Itô formula on $\varphi(t)|\mathcal{P}^l_K\xi_t|^2$ as in the previous section, taking the expectation with respect to $P_{\mu_\star}$ and then getting the limit as $K\uparrow\infty$ yields the following representation formula for $\iota(\mu_\star)$,

$$\iota(\mu_\star) = \lim_{K\uparrow\infty} \mathbb{E}^{\mu_\star}[\langle \mathcal{P}^l_K x, \mathcal{P}^l_K B(x,x)\rangle].$$

Since $x = \mathcal{P}^l_K x + \mathcal{P}^h_K x$,

$$\begin{aligned}\langle \mathcal{P}^l_K x, \mathcal{P}^l_K B(x,x)\rangle &= \langle \mathcal{P}^l_K x, B(x, \mathcal{P}^l_K x)\rangle + \langle \mathcal{P}^l_K x, B(x, \mathcal{P}^h_K x)\rangle \\ &= \langle \mathcal{P}^l_K x, B(x, \mathcal{P}^h_K x)\rangle\end{aligned}$$

as $\langle \mathcal{P}^l_K x, B(x, \mathcal{P}^l_K x)\rangle$ is the sum of a finite number of terms (so we can use the anti-symmetric property of the non-linear term without convergence issues). In conclusion,

$$\iota(\mu_\star) = \lim_{K\uparrow\infty} \mathbb{E}^{\mu_\star}[\langle \mathcal{P}^l_K x, B(x, \mathcal{P}^h_K x)\rangle] = \lim_{K\uparrow\infty} \mathbb{E}^{\mu_\star}\Bigg[\sum_{\substack{l+m=k\\ |k|_\infty\le K,\\ |m|_\infty > K}} (x_m\cdot\overline{x_k})(m\cdot x_l)\Bigg].$$

Following Frisch [14, Sect. 6.2], the last term we have obtained in the formula above is the *energy flux* through wave-number K and represents the energy transferred from the scales up to K to smaller scales.

From the previous section we know that $\iota(\mu_\star)\ge 0$, this is a consequence of property [M3] of Definition 2.4. From a mathematical point of view, existence of invariant measures with $\iota(\mu_\star) > 0$ would be an evidence for loss of regularity and, in turn, for blow-up. From a physical point of view, the energy flux through wave-numbers should converge to zero—hence, again we would expect $\iota(\mu_\star) = 0$—as the energy should flow through modes essentially only in the inertial range (we refer again to Frisch [14]).

Proposition 4.11 *We have*

1. *the map $\mu_\star \mapsto \varepsilon(\mu_\star)$ has a smallest element in $\mathcal{I}$* (solution of largest mean inertial dissipation),
2. *if*

$$\lim_{R\uparrow\infty} \sup_{\mu\in\mathcal{I}_m} \mathbb{E}^\mu[(|x|^2_V - R)\mathbb{1}_{\{|x|^2_V > R\}}] = 0,$$

then there is $\mu_\star\in\mathcal{I}$ such that $\varepsilon(\mu_\star) = \sup_{\mu\in\mathcal{I}_m}\varepsilon(\mu_\star)$ (solution of smallest mean inertial dissipation).

Proof The first part follows easily as $\mathcal{I}$ is compact (see Remark 4.2) and $\mu\to\varepsilon(\mu)$ is lower semi-continuous for the topology with respect to which $\mathcal{I}$ is compact. As it regards the second part, we know by Corollary 3.2 that $\mathcal{I}_m$ is relatively compact on $C_b(V)$. Hence, if M is the largest value attained by ε on $\mathcal{I}_m$ and $(\mu_n)_{n\in\mathbf{N}}$ is a maximizing sequence, say

$\varepsilon(\mu_n) \geq M - \frac{1}{n}$, by compactness there is μ_∞ such that, up to a sub-sequence, $\mu_n \to \mu_\infty$. Now

$$\begin{aligned}\varepsilon_R(\mu_n) &= \mathbb{E}^{\mu_n}[|x|_V^2 \mathbb{1}_{\{|x|_V^2 \leq R\}}] + R\mu_n[|x|_V^2 > R] \\ &\geq M - \frac{1}{n} - \sup_{\mu \in \mathcal{I}} \mathbb{E}^\mu[(|x|_V^2 - R)\mathbb{1}_{\{|x|_V^2 > R\}}],\end{aligned}$$

where $\varepsilon_R(\mu) = \mathbb{E}^\mu[|x|_V^2 \wedge R]$. As $n \uparrow \infty$, by continuity $\varepsilon_R(\mu_n) \to \varepsilon_R(\mu_\infty)$, so

$$\varepsilon(\mu_\infty) \geq \varepsilon_R(\mu_\infty) \geq M - \sup_{\mu \in \mathcal{I}} \int |x|_V^2 \mathbb{1}_{\{|x|_V^2 > R\}}\, \mu(dx).$$

As $R \to \infty$, it follows that $\varepsilon(\mu_\infty) \geq M$, hence $\varepsilon(\mu_\infty) = M$. □

We have not been able yet to prove the condition given in item (2) of previous proposition.

We also remark that such measures of largest and smallest mean inertial dissipation may not be unique, as both functionals $\varepsilon(\cdot)$ and $\iota(\cdot)$ are translations-invariant (see Section 4.2).

Appendix A: Analysis of the Mildly Regular Approximated Problem

Let $R \geq 1$ and let $\chi_R : [0, \infty] \to [0, 1]$ be a non-increasing C^∞ function such that $\chi_R \equiv 1$ on $[0, \frac{3}{2}R]$, $\chi_R \equiv 0$ on $[2R, \infty)$ and there is $c > 0$ such that $|\chi_R'| \leq \frac{c}{R}$ (see Fig. 1).

Given a value $\varepsilon_0 \in (0, \frac{1}{4}]$, we consider the following problem in $\mathcal{V}_{\varepsilon_0}$,

$$\begin{cases} d\tilde{u}^{(\varepsilon_0,R)} + \nu A\tilde{u}^{(\varepsilon_0,R)} + \chi_R(|\tilde{u}^{(\varepsilon_0,R)}|_{\mathcal{V}_{\varepsilon_0}}^2) B(\tilde{u}^{(\varepsilon_0,R)}, \tilde{u}^{(\varepsilon_0,R)}) = \mathcal{Q}^{\frac{1}{2}} dW, \\ \operatorname{div} \tilde{u}^{(\varepsilon_0,R)} = 0. \end{cases} \tag{A.1}$$

Let $\tau^{(\varepsilon_0,R)} : \Omega \to [0, \infty)$ be defined as

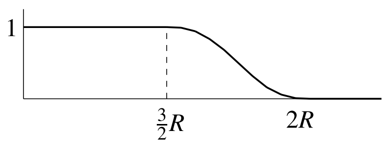
$$\tau^{(\varepsilon_0,R)}(\omega) = \inf\left\{ t \geq 0 : |\omega(t)|_{\mathcal{V}_{\varepsilon_0}}^2 \geq \frac{3}{2}R \right\}$$

(and $\tau^{(\varepsilon_0,R)}(\omega) = \infty$ if the above set is empty). The main aim of this section is to analyse the solutions to the above problems and their connections to the original Navier-Stokes equations (1.2).

Before turning to the results on the regularized problem (A.1), we remark that in the proof of all results of this section we shall use the splitting $\tilde{u}^{(\varepsilon_0,R)} = \tilde{v}^{(\varepsilon_0,R)} + z$, where z solves the following linear Stokes problem

$$\begin{cases} dz + Az\, dt = \mathcal{Q}^{\frac{1}{2}}\, dW, \\ z(0) = 0, \end{cases} \tag{A.2}$$

Fig. 1 The cut-off function χ_R



and so $\tilde{v}^{(\varepsilon_0,R)}$ solves the following equation with random coefficients

$$\frac{d}{dt}\tilde{v}^{(\varepsilon_0,R)} + \nu A\tilde{v}^{(\varepsilon_0,R)} + \chi_R(|\tilde{u}^{(\varepsilon_0,R)}|^2_{\mathcal{V}_{\varepsilon_0}})B(\tilde{u}^{(\varepsilon_0,R)}, \tilde{u}^{(\varepsilon_0,R)}) = 0. \tag{A.3}$$

A.1 The Weak-Strong Uniqueness Principle

We first extend the weak-strong uniqueness principle stated in [11, Theorem 5.4] to the above problem (A.1). This is the content of the following result.

Theorem A.1 *Assume condition* [A2] *of Assumption* 1.1 *and let* $\varepsilon_0 \in (0, \frac{1}{4}]$ *with* $\varepsilon_0 < 2\alpha_0$. *Then, for every* $x \in \mathcal{V}_{\varepsilon_0}$ *equation* (*A*.1) *has a unique martingale solution* $P_x^{(\varepsilon_0,R)}$, *with*

$$P_x^{(\varepsilon_0,R)}[C([0,\infty);\mathcal{V}_{\varepsilon_0})] = 1. \tag{A.4}$$

Moreover, the following statements hold.

1. (weak-strong uniqueness) *On the interval* $[0, \tau^{(\varepsilon_0,R)}]$, *the probability measure* $P_x^{(\varepsilon_0,R)}$ *coincides with any martingale solution* P_x *of the original stochastic Navier-Stokes equations* (1.2), *namely*

$$\mathbb{E}^{P_x^{(\varepsilon_0,R)}}\left[\varphi(\xi_t)\mathbb{1}_{\{\tau^{(\varepsilon_0,R)}\geq t\}}\right] = \mathbb{E}^{P_x}\left[\varphi(\xi_t)\mathbb{1}_{\{\tau^{(\varepsilon_0,R)}\geq t\}}\right]$$

   for every $t \geq 0$ *and every bounded measurable* $\varphi : H \to \mathbf{R}$.
2. (blow-up estimate) *There are* $c_0 > 0$, $c_1 > 0$ *and* $c_2 > 0$, *depending only on* ε_0, *such that for every* $x \in \mathcal{V}_{\varepsilon_0}$ *with* $|x|^2_{\mathcal{V}_{\varepsilon_0}} \leq \frac{R}{2}$,

$$P_x^{(\varepsilon_0,R)}[\tau^{(\varepsilon_0,R)} \leq \delta] \leq c_1 \mathrm{e}^{-c_2\frac{R^2}{\delta}} \tag{A.5}$$

   for every $\delta \leq c_0 R^{-\frac{1}{2\varepsilon_0}}$.

Proof The proof is developed in four steps, which are contained in the following subsections. More precisely, in the first step we prove existence of solutions for problem (A.1), while in the second step we prove uniqueness. The weak-strong uniqueness principle is then proved in the third step and the blow-up estimate (A.5) is given in the fourth step.
Step 1: Existence. We only show the key estimate for existence. Let z be the solution to the linear Stokes problem (A.2) and consider $\tilde{v}^{(\varepsilon_0,R)}$ as above. The usual energy estimate provides (here we use $\tilde{u} = \tilde{u}^{(\varepsilon_0,R)}$ and $\tilde{v} = \tilde{v}^{(\varepsilon_0,R)}$ for brevity)

$$\begin{aligned}
\frac{d}{dt}|\tilde{v}|^2_{\mathcal{V}_{\varepsilon_0}} + 2\nu\|\tilde{v}\|^2_{\mathcal{V}_{\varepsilon_0}} &\leq 2\chi_R(|\tilde{u}|^2_{\mathcal{V}_{\varepsilon_0}}|)\langle A^{\varepsilon_0-\frac{1}{4}}B(\tilde{u},\tilde{u}), A^{\frac{3}{4}+\varepsilon_0}\tilde{v}\rangle_H \\
&\leq c\chi_R(|\tilde{u}|^2_{\mathcal{V}_{\varepsilon_0}}|)\|\tilde{v}\|_{\mathcal{V}_{\varepsilon_0}}|A^{\theta(\varepsilon_0)}\tilde{u}|^2 \\
&\leq c\chi_R(|\tilde{u}|^2_{\mathcal{V}_{\varepsilon_0}}|)\|\tilde{v}\|_{\mathcal{V}_{\varepsilon_0}}(|A^{\theta(\varepsilon_0)}z|^2 + |\tilde{v}|^{1+2\varepsilon_0}_{\mathcal{V}_{\varepsilon_0}}\|\tilde{v}\|^{1-2\varepsilon_0}_{\mathcal{V}_{\varepsilon_0}}) \\
&\leq \nu\|\tilde{v}\|^2_{\mathcal{V}_{\varepsilon_0}} + c(|A^{\theta(\varepsilon_0)}z|^4 + |z|^{2+\frac{1}{\varepsilon_0}}_{\mathcal{V}_{\varepsilon_0}} + R^{1+\frac{1}{2\varepsilon_0}}),
\end{aligned} \tag{A.6}$$

where we have used interpolation inequalities and Lemma 2.2, with $\alpha = \varepsilon_0$. Notice that, by the choice of ε_0 with respect to α_0, $|A^{\theta(\varepsilon_0)}z|^2_H$ has exponential moments.

In order to show (A.4), we show an a-priori estimate for the derivative in time $\frac{d}{dt}\tilde{v}^{(\varepsilon_0,R)}$ in $L^2(0,T;D(A^{-(\frac{1}{4}-\varepsilon_0)}))$, for all $T>0$. The continuity of $\tilde{u}^{(\varepsilon_0,R)}$ then follows from this fact and continuity of z. The a-priori estimate follows by multiplying the equations by $\frac{d}{dt}A^{2\varepsilon_0-\frac{1}{2}}\tilde{v}^{(\varepsilon_0,R)}$,

$$\begin{aligned}|A^{\varepsilon_0-\frac{1}{4}}\dot{\tilde{v}}|^2+\frac{\nu}{2}\frac{d}{dt}|\tilde{v}|^2_{\mathcal{V}_{\varepsilon_0}} &\leq \chi_R(|\tilde{u}|^2_{\mathcal{V}_{\varepsilon_0}}|)|\langle A^{\varepsilon_0-\frac{1}{4}}B(\tilde{u},\tilde{u}),A^{\varepsilon_0-\frac{1}{4}}\dot{\tilde{v}}\rangle_H\\ &\leq \frac{1}{2}|A^{\varepsilon_0-\frac{1}{4}}\dot{\tilde{v}}|^2+c_4\big(|A^{\theta(\varepsilon_0)}z|^4+R^{1+2\varepsilon_0}\|\tilde{v}\|^{2-4\varepsilon_0}_{\mathcal{V}_{\varepsilon_0}}\big),\end{aligned}$$

where we have used the same estimates as in (A.6) (and again $\tilde{u}=\tilde{u}^{(\varepsilon_0,R)}$ and $\tilde{v}=\tilde{v}^{(\varepsilon_0,R)}$ for brevity).

Step 2: Uniqueness. Let $\tilde{u}_1$, $\tilde{u}_2$ be two solutions of (A.1) starting at the same initial condition and set $w=\tilde{u}_1-\tilde{u}_2$. The new process w solves the following equation with random coefficients,

$$\begin{aligned}\dot{w}+\nu Aw &= \chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})B(\tilde{u}_2,\tilde{u}_2)-\chi_R(|\tilde{u}_1|^2_{\mathcal{V}_{\varepsilon_0}})B(\tilde{u}_1,\tilde{u}_1)\\ &= \big[\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})-\chi_R(|\tilde{u}_1|^2_{\mathcal{V}_{\varepsilon_0}})\big]B(\tilde{u}_1,\tilde{u}_2)+\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})B(w,\tilde{u}_2)\\ &\quad +\chi_R(|\tilde{u}_1|^2_{\mathcal{V}_{\varepsilon_0}})B(\tilde{u}_1,w)\end{aligned}$$

and so

$$\begin{aligned}\frac{d}{dt}|w|^2_H+2\nu|w|^2_V &= 2\big[\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})-\chi_R(|\tilde{u}_1|^2_{\mathcal{V}_{\varepsilon_0}})\big]\langle w,B(\tilde{u}_1,\tilde{u}_2)\rangle_H\\ &\quad +2\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})\langle w,B(w,\tilde{u}_2)\rangle_H\\ &= \boxed{1}+\boxed{2}.\end{aligned}$$

Next, we estimate the two terms. In order to estimate the first term, we first remark that

$$|\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})-\chi_R(|\tilde{u}_1|^2_{\mathcal{V}_{\varepsilon_0}})|\leq\frac{c}{\sqrt{R}}|w|_{\mathcal{V}_{\varepsilon_0}}[\mathbb{1}_{[0,2R]}(|\tilde{u}_1|^2_{\mathcal{V}_{\varepsilon_0}})+\mathbb{1}_{[0,2R]}(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})].$$

By using the above inequality, Lemma 2.1 (with $\alpha_1=\frac{1}{2}-\varepsilon_0$, $\alpha_2=0$ and $\alpha_3=\frac{1}{4}+\varepsilon_0$) and interpolation and Young's inequalities, we get[3]

$$\begin{aligned}\boxed{1} &\leq |\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})-\chi_R(|\tilde{u}_1|^2_{\mathcal{V}_{\varepsilon_0}})\big||\langle B(\tilde{u}_1,w),\tilde{u}_2\rangle_H|\\ &\leq \frac{c}{\sqrt{R}}\sqrt{2R}|w|_{\mathcal{V}_{\varepsilon_0}}|w|_V(|A^{\frac{1}{2}-\varepsilon_0}\tilde{u}_1|_H+|A^{\frac{1}{2}-\varepsilon_0}\tilde{u}_2|_H)\\ &\leq c|w|_H^{\frac{1}{2}-2\varepsilon_0}|w|_V^{\frac{3}{2}+2\varepsilon_0}(|A^{\frac{1}{2}-\varepsilon_0}\tilde{u}_1|_H+|A^{\frac{1}{2}-\varepsilon_0}\tilde{u}_2|_H)\end{aligned}$$

[3]The inequality has to be slightly modified if $\varepsilon_0=\frac{1}{4}$. In such a case we use Lemma 2.1 with $\alpha_1=\frac{1}{2}$, $\alpha_2=0$, $\alpha_3=\frac{1}{4}$,

$$|\langle B(\tilde{u}_1,\tilde{u}_2),w\rangle_H|\leq c|A^{\frac{1}{4}}w|_H|\tilde{u}_1|_V|\tilde{u}_2|_V.$$

$$\le \frac{\nu}{2}|w|_V^2 + c|w|_H^2(|A^{\frac{1}{2}-\varepsilon_0}\tilde{u}_1|_H + |A^{\frac{1}{2}-\varepsilon_0}\tilde{u}_2|_H)^{\frac{4}{1-4\varepsilon_0}}.$$

For the second term, we use Lemma 2.1 with $\alpha_1 = \frac{1}{4} + \varepsilon_0$, $\alpha_2 = 0$ and $\alpha_3 = \frac{1}{2} - \varepsilon_0$, and interpolation and Young's inequalities,

$$\begin{aligned}
\boxed{2} &= 2\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})|\langle \tilde{u}_2, B(w,w)\rangle_H \\
&\le 2c\chi_R(|\tilde{u}_2|^2_{\mathcal{V}_{\varepsilon_0}})|\tilde{u}_2|_{\mathcal{V}_{\varepsilon_0}}|w|_V\,|A^{\frac{1}{2}-\varepsilon_0}w|_H \\
&\le 2cR|w|_H^{2\varepsilon_0}|w|_V^{2(1-\varepsilon_0)} \\
&\le \frac{1}{2}|w|_V^2 + cR^{\frac{1}{\varepsilon_0}}|w|_H^2.
\end{aligned} \tag{A.7}$$

Finally, Gronwall's lemma implies that $w \equiv 0$, since $w(0) = 0$.
Step 3: Weak-strong uniqueness. The proof works exactly as in [11, Theorem 5.12], we give a short account for the sake of completeness. The proof is developed in the following steps.

1. The energy balance of $\tilde{w} = \tilde{u}^{(\varepsilon_0,R)} - \xi$, given by

$$\begin{aligned}
\tilde{\mathcal{E}}_t &= |\tilde{w}(t)|_H^2 + 2\nu\int_0^t |\tilde{w}(s)|_V^2\,ds + \\
&\quad + 2\int_0^t \Big(\langle \chi_R(|\tilde{u}^{(\varepsilon_0,R)}|^2_{\mathcal{V}_{\varepsilon_0}})B(\tilde{u}^{(\varepsilon_0,R)}, \tilde{u}^{(\varepsilon_0,R)}) - B(\xi,\xi), \tilde{w}\rangle_H\Big)\,ds
\end{aligned}$$

   is an a. s. super-martingale under P_x.
2. $\tau^{(\varepsilon_0,R)}$ is a stopping time with respect to the filtration $(\mathcal{B}_t)_{t\ge 0}$.
3. The stopped process $(\tilde{\mathcal{E}}_{t\wedge\tau^{(\varepsilon_0,R)}})_{t\ge 0}$ is again an a. s. super-martingale.
4. The previous step implies the conclusion.

All the above steps can be carried out exactly as in the proof of Theorem 5.12 of [11] (the key point is that $\tilde{u}^{(\varepsilon_0,R)}$ is continuous in time with values in $\mathcal{V}_{\varepsilon_0}$ with probability one). The only difference is in the last step, where the estimate of the non-linearity needs to be replaced by the following estimate,

$$\begin{aligned}
&\langle \tilde{w}, B(\tilde{u}^{(\varepsilon_0,R)}, \tilde{u}^{(\varepsilon_0,R)}) - B(\xi,\xi)\rangle_H \\
&\quad = -\langle B(\tilde{w},\tilde{w}), \tilde{u}^{(\varepsilon_0,R)}\rangle_H \le c|\tilde{u}^{(\varepsilon_0,R)}|_{\mathcal{V}_{\varepsilon_0}}|\tilde{w}|_V|A^{\frac{1}{2}-\varepsilon_0}\tilde{w}|_H \\
&\quad \le c\sqrt{R}|\tilde{w}|_V^{2(1-\varepsilon_0)}|\tilde{w}|_H^{2\varepsilon_0} \le \nu|\tilde{w}|_V^2 + c\nu^{-\frac{\varepsilon_0}{1-\varepsilon_0}}R^{\frac{1}{2\varepsilon_0}}|\tilde{w}|_H^2,
\end{aligned}$$

which follows from Lemma 2.1 (with $\alpha_1 = \varepsilon_0 + \frac{1}{4}$, $\alpha_2 = 0$ and $\alpha_3 = \frac{1}{2} - \varepsilon_0$) and interpolation and Young's inequalities. Finally, the above estimate can be obtained as in (A.7).
Step 4: The blow-up estimate. Fix $x \in \mathcal{V}_{\varepsilon_0}$ with $|x|^2_{\mathcal{V}_{\varepsilon_0}} \le \frac{R}{2}$ and $\delta > 0$. Set $\Theta_\delta = \sup_{s\in[0,\delta]}|A^{\theta(\varepsilon_0)}z|^2$, then, by slightly modifying inequality (A.6), we get

$$\frac{d}{dt}|\tilde{v}^{(\varepsilon_0,R)}|^2_{\mathcal{V}_{\varepsilon_0}} \le c(|A^{\theta(\varepsilon_0)}z|_H^4 + |\tilde{v}^{(\varepsilon_0,R)}|_{\mathcal{V}_{\varepsilon_0}}^{2+\frac{1}{\varepsilon_0}}) \le c\left(\frac{1}{3} + \Theta_\delta^2 + |\tilde{v}^{(\varepsilon_0,R)}|^2_{\mathcal{V}_{\varepsilon_0}}\right)^{\frac{2\varepsilon_0+1}{2\varepsilon_0}},$$

where $\tilde{v}^{(\varepsilon_0,R)} = \tilde{u}^{(\varepsilon_0,R)} - z$ has been defined in (A.3). Hence, if we set $\varphi(t) = \frac{1}{3} + \Theta_\delta^2 + |\tilde{v}^{(\varepsilon_0,R)}|^2_{\mathcal{V}_{\varepsilon_0}}$, we end up with a differential inequality that, once solved, gives

$$\varphi(t) \le \varphi(0)\big(1 - c\delta\varphi(0)^{\frac{1}{2\varepsilon_0}}\big)^{-2\varepsilon_0}.$$

From this, it is easy to show that there is a suitable constant $c_0 = c_0(\varepsilon_0) > 0$ such that $|\tilde{u}^{\varepsilon_0,R}(s)|^2_{\mathcal{V}_{\varepsilon_0}} < \frac{3}{2}R$ for every $s \le \delta$, when $\Theta_\delta \le \frac{R}{2}$ and $\delta \le c_0 R^{-\frac{1}{2\varepsilon_0}}$.

In conclusion, if $\delta \le c_0 R^{-\frac{1}{2\varepsilon_0}}$ and $|x|^2_{\mathcal{V}_{\varepsilon_0}} \le \frac{R}{2}$, then

$$\Theta_\delta \le \frac{R}{2} \quad\Longrightarrow\quad |\tilde{u}^{(\varepsilon_0,R)}(t)|^2_{\mathcal{V}_{\varepsilon_0}} < \frac{3}{2}R \quad \text{for } t \le \delta \quad\Longrightarrow\quad \tau^{(\varepsilon_0,R)} > \delta$$

and so

$$P_x^{(\varepsilon_0,R)}[\tau^{(\varepsilon_0,R)} \le \delta] \le P_x^{(\varepsilon_0,R)}\left[\Theta_\delta > \frac{R}{2}\right] \le c_1 \mathrm{e}^{-c_2 \frac{R^2}{\delta}}$$

with constants $c_1 = c_1(\varepsilon_0) > 0$ and $c_2 = c_2(\varepsilon_0) > 0$ depending only on ε_0 and the last inequality follows easily as in Proposition 15 of [12] (which in turns is a consequence of Proposition 2.16 of Da Prato & Zabczyk [3]). □

A.2 Moments of Norms of Stronger Regularity

The proof of the following theorem is based on an inequality taken from Temam [21, Sect. 4.3, Part I] (see also Odasso [17]).

Theorem A.2 *Under condition* [A2] *of Assumption* 1.1, *for every* $\varepsilon_0 \in (0, \frac{1}{4}]$, *with* $\varepsilon_0 < 2\alpha_0$, *there are* $\delta = \delta(\varepsilon_0, \alpha_0) > 0$ *and* $\gamma = \gamma(\varepsilon_0, \alpha_0) > 0$ *such that*

$$\mathbb{E}\Big[\int_0^t |A^\delta \tilde{u}^{(\varepsilon_0,R)}|^{2\gamma}_{\mathcal{W}_{\alpha_0}}\, ds\Big] \le C[1 + t + |x|^2_H], \tag{A.8}$$

where $\tilde{u}^{(\varepsilon_0,R)}$ *is the solution to problem* (A.1) *starting at* $x \in \mathcal{V}_{\varepsilon_0}$ *and the value of* C *is independent of both* x *and* R.

Proof If $\alpha_0 \le \frac{1}{4}$, we choose $\varepsilon_0 < 2\alpha_0$ (such condition is useless for all other values of α_0). The noise is not regular enough to let us work directly on $\tilde{u}^{(\varepsilon_0,R)}$, so we rely, as in the proof of the previous theorem, on $\tilde{v}^{(\varepsilon_0,R)} = \tilde{u}^{(\varepsilon_0,R)} - z$. Let $p = \frac{1}{2} - \varepsilon_0$ (the value of p could be slightly improved, but it is beyond our needs) and compute

$$\frac{d}{dt}[(1 + |\tilde{v}|^2_{\mathcal{V}_{\varepsilon_0}})^{-p}] = 2p\frac{\|\tilde{v}\|^2_{\mathcal{V}_{\varepsilon_0}}}{(1 + |\tilde{v}|^2_{\mathcal{V}_{\varepsilon_0}})^{p+1}} - 2p\chi_R(|\tilde{u}|^2_{\mathcal{V}_{\varepsilon_0}})\frac{\langle A^{\frac{1}{2}+2\varepsilon_0}\tilde{v}, B(\tilde{u},\tilde{u})\rangle_H}{(1 + |\tilde{v}|^2_{\mathcal{V}_{\varepsilon_0}})^{p+1}},$$

where we have set $\tilde{u} = \tilde{u}^{(\varepsilon_0,R)}$ and $\tilde{v} = \tilde{v}^{(\varepsilon_0,R)}$. The non-linear term can be estimated as in (A.6) to get

$$\langle A^{\varepsilon_0+\frac{3}{4}}\tilde{v}, A^{\varepsilon_0-\frac{1}{4}}B(\tilde{u},\tilde{u})\rangle_H \le \frac{1}{2}\|\tilde{v}\|^2_{\mathcal{V}_{\varepsilon_0}} + c\big(|A^{\theta(\varepsilon_0)}z|^4_H + |\tilde{v}|^{2+\frac{1}{\varepsilon_0}}_{\mathcal{V}_{\varepsilon_0}}\big)$$

and so

$$\int_0^t \frac{\|\tilde v\|^2_{\mathcal V_{\varepsilon_0}}}{(1+|\tilde v|^2_{\mathcal V_{\varepsilon_0}})^{p+1}}\,ds \le 1 + c\int_0^t |A^{\theta(\varepsilon_0)}z|^4\,ds + c\int_0^t |\tilde v|^2_{\mathcal V_{\varepsilon_0}}\,ds.$$

The term in z is plain (see for example Da Prato & Zabczyk [3]), while the term in $|\tilde v|_{\mathcal V_{\varepsilon_0}}$ can be estimated by means of the energy inequality [M3] of Definition 2.4. Finally, in order to prove (A.8), we use again the energy balance, since by Young's inequality

$$\|\tilde v\|^{2\gamma}_{\mathcal V_{\varepsilon_0}} \le c\Big[\frac{\|\tilde v\|^2_{\mathcal V_{\varepsilon_0}}}{(1+|\tilde v|^2_{\mathcal V_{\varepsilon_0}})^{p+1}} + (1+|\tilde v|^2_{\mathcal V_{\varepsilon_0}})\Big]$$

if γ is chosen properly, depending on p.

Next, if $\alpha_0 > \frac14$, fix $\alpha \in (\frac14, \alpha_0)$ and $\varepsilon_0 \in (0, \frac14]$ and choose $p > 0$ (whose value will be fixed in dependence of α). We apply Itô formula to the function $(1+|A^\alpha \tilde u^{(\varepsilon_0,R)}(t)|^2_H)^{-p}$, to get

$$\begin{aligned}
&\frac{1}{(1+|A^\alpha\tilde u(t)|^2_H)^p} - \frac{1}{(1+|A^\alpha x|^2_H)^p}\\
&= 2p\int_0^t \frac{|A^{\alpha+\frac12}\tilde u|^2_H}{(1+|A^\alpha\tilde u|^2_H)^{p+1}}\,ds + 2p\int_0^t \chi_R(|\tilde u|^2_{\mathcal V_{\varepsilon_0}})\frac{\langle A^{\alpha+\frac12}\tilde u, A^{\alpha-\frac12}B(\tilde u,\tilde u)\rangle_H}{(1+|A^\alpha\tilde u|^2_H)^{p+1}}\,ds\\
&\quad - 2p\int_0^t \frac{\langle A^\alpha\tilde u, A^\alpha \mathcal Q^{\frac12}dW_s\rangle_H}{(1+|A^\alpha\tilde u|^2_H)^{p+1}}\,ds - p\int_0^t \frac{\sigma_\alpha^2}{(1+|A^\alpha\tilde u|^2_H)^{p+1}}\,ds\\
&\quad + 2p(p+1)\int_0^t \frac{|A^\alpha\mathcal Q^{\frac12}\tilde u|^2_H}{(1+|A^\alpha\tilde u|^2_H)^{p+2}}\,ds,
\end{aligned}$$

and we have set again $\tilde u = \tilde u^{(\varepsilon_0,R)}$. The non-linear part is estimated with Lemma 2.2, interpolation and Young's inequalities,

$$\begin{aligned}
\langle A^{\alpha+\frac12}\tilde u, A^{\alpha-\frac12}B(\tilde u,\tilde u)\rangle_H &\le C|A^{\alpha+\frac12}\tilde u|_H |A^{\theta(\alpha-\frac14)}\tilde u|^2_H\\
&\le C|A^{\alpha+\frac12}\tilde u|_H |A^\alpha\tilde u|_H^{2\alpha+\frac12}|A^{\alpha+\frac12}\tilde u|_H^{\frac32-2\alpha}\\
&\le \frac12|A^{\alpha+\frac12}\tilde u|^2_H + C|A^\alpha\tilde u|_H^{2\frac{4\alpha+1}{4\alpha-1}}.
\end{aligned}$$

If $\alpha \le \frac14+\varepsilon_0$, $\alpha+\frac12 > \theta(\alpha_0)$ and one already knows that some power of $|A^\alpha u|$ has finite moment, then one can proceed as in the previous case $\alpha_0 \le \frac14$. Otherwise, as in Temam [21], one can iterate the same procedure using $\alpha - \frac12$ instead of α, until the above conditions are satisfied. □

A.3 An Estimate of the Return Time to a Ball

The aim of this section is to verify that the probability of hitting a ball (in a smooth norm) can be uniformly bounded from below for all initial conditions in a given ball.

Lemma A.3 *Assume condition* [A3] *from Assumption 1.1. Then one can choose $\varepsilon_0 \in (0,\frac14]$ with $\varepsilon_0 < 2\alpha_0$ such that there are $\theta' < \alpha_0+\frac12$, $\theta'' > \theta(\alpha_0)$ and a suitable constant $c>0$, and the following statement holds.*

For every $R \geq 1$ there are values $T_0 = T_0(R)$ and $K = K(R)$ such that for every $x \in \mathcal{V}_{\varepsilon_0}$,

$$[\mathrm{A}] \begin{cases} |x|^2_{\mathcal{V}_{\varepsilon_0}} \leq R, \\ \sup_{t\in[0,T_0]} |A^{\theta'} z(t)|^2_H \leq R \\ T_0 < cR^{-\frac{1}{2\varepsilon_0}}, \end{cases} \qquad \Longrightarrow \qquad [\mathrm{B}] \begin{cases} \tau^{\varepsilon_0,3R} > T_0, \\ |A^{\theta''} \tilde{u}^{\varepsilon_0,3R}(T_0)|^2_H \leq K, \end{cases}$$

where z is the solution to the linear problem (A.2) .

Proof We choose $\varepsilon_0 = \frac{1}{4}$ and we set, for brevity, $\tilde{u} = \tilde{u}^{\varepsilon_0,3R}$ and $\tilde{v} = \tilde{u} - z$. The first part of statement [B] follows as in the proof of (A.5), if the constant c is chosen accordingly. So, for every $t \in [0, T_0]$, we know that $|\tilde{u}(t)|^2_V \leq 3R$. In particular, using (A.6) and the second statement of [A], it follows that there is a constant $K_0 = K_0(R)$ such that

$$\sup_{[0,T_0]} |\tilde{v}(t)|^2_V + \int_0^{T_0} |A\tilde{v}(s)|^2_H\, ds \leq K_0(R). \tag{A.9}$$

We next prove the second statement of [B].

Step 1. We first consider $\alpha_0 \in (\frac{1}{6}, \frac{1}{2}]$ and we choose $\delta \in (0, \frac{1}{4})$ so that $\frac{\alpha_0}{2} < \delta < \delta + \frac{1}{4}$ (this condition ensures that $\theta(\alpha_0) < \frac{1}{2} + \delta$ and $\theta(\delta + \frac{1}{4}) < \alpha_0 + \frac{1}{2}$). In the case $\alpha = \frac{1}{2}$ we simply choose a value $\delta \in (\frac{1}{4}, \frac{1}{2})$.

Step 2. *For all $\omega \in \Omega$ satisfying* [A], *there is $t_0 = t_0(\omega) \in (0, T_0)$ such that $|A\tilde{v}(t_0)|^2_H \leq 2K_0$*. Indeed, from (A.9) it follows that the set $\{t \in (0, T_0) : |A\tilde{v}(t)_H|^2 \leq 2K_0\}$ has Lebesgue measure at least $\frac{T_0}{2}$, and in particular is not empty.

Step 3. *There is $K_1 = K_1(R)$ such that for all $\omega \in \Omega$ satisfying* [A], $|A^{\frac{1}{2}+\delta}\tilde{u}(T_0)|^2_H \leq K_1$. We use Lemma 2.2 (with $\alpha = \delta + \frac{1}{4}$), interpolation of $D(A^{\theta(\delta+\frac{1}{4})})$ between V and $D(A^{1+\delta})$ and Young's inequality to obtain the following estimate,

$$\begin{aligned} \frac{d}{dt}|A^{\frac{1}{2}+\delta}\tilde{v}|^2_H + 2\nu|A^{1+\delta}\tilde{v}|^2_H &= 2\langle A^{1+2\delta}\tilde{v}, B(\tilde{u},\tilde{u})\rangle \\ &\leq c|A^{1+\delta}\tilde{v}|_H |A^{\theta(\delta+\frac{1}{4})}\tilde{u}|^2_H \\ &\leq \nu|A^{1+\delta}\tilde{v}|_H + C\big(|A^{\theta(\delta+\frac{1}{4})}z|^4_H + |\tilde{v}|^{6+8\delta}_V\big). \end{aligned}$$

Since $\tilde{v}$ is bounded in V, the claim easily follows. In the special case $\alpha_0 = \frac{1}{2}$ one can proceed analogously.

Step 4. We choose then $\theta' = \theta(\delta + \frac{1}{4})$ and $\theta'' = \delta + \frac{1}{2}$ and the second statement of [B] follows.

Step 5. If $\alpha_0 > \frac{1}{2}$ we iterate the above procedure as in the proof of Lemma C.1 of [11], using the two inequalities

[i1] if $m \in \mathbf{N}$, $m \geq 1$, there are an integer p_m and $C_m > 0$ such that

$$\frac{d}{dt}|A^{\frac{m}{2}}\tilde{v}|^2_H + \nu|A^{\frac{m+1}{2}}\tilde{v}|^2_H \leq C_m(1 + |\tilde{v}|_V + |A^{\frac{m}{2}}z|_H)^{p_m},$$

[i2] if $\kappa \geq \frac{1}{2}$ and $\beta \in [0, \frac{1}{2})$, there are $C_{\kappa,\beta} > 0$ and $a_\beta > 0$ such that

$$\begin{aligned} \frac{d}{dt}|A^{\kappa+\beta}\tilde{v}|^2_H + \nu|A^{\kappa+\beta+\frac{1}{2}}\tilde{v}|^2_H \leq C_{\kappa,\beta}\big[&|A^{\kappa+\beta}z|^4_H + (|A^{\kappa+\frac{1}{2}}\tilde{v}|^2_H|A^{\kappa}\tilde{v}|^{a_\beta} \\ &+ |A^{\kappa+\beta}z|^2_H)|A^{\kappa+\beta}\tilde{v}|^2_H\big], \end{aligned}$$

whose proof can be found in the proof of that same lemma. □

Acknowledgements The author wishes to thank D. Blömker and F. Flandoli for the several useful conversations and their helpful comments. The author is also grateful to A. Debussche for having pointed out the inequality used in the proof of Theorem A.2.

References

1. Blömker, D., Flandoli, F., Romito, M.: Markovianity and ergodicity for a surface growth PDE. Preprint (2006). Available at http://arXiv.org/abs/math.PR/0611021
2. Da Prato, G., Debussche, A.: Ergodicity for the 3D stochastic Navier-Stokes equations. J. Math. Pures Appl. (9) **82**(8), 877–947 (2003)
3. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge University Press, Cambridge (1992)
4. Da Prato, G., Zabczyk, J.: Ergodicity for Infinite-Dimensional Systems. London Mathematical Society Lecture Note Series, vol. 229. Cambridge University Press, Cambridge (1996)
5. Debussche, A., Odasso, C.: Markov solutions for the 3D stochastic Navier-Stokes equations with state dependent noise. J. Evol. Equ. **6**(2), 305–324 (2006)
6. Dellacherie, C., Meyer, P.-A.: Theory of martingales. In: Probabilities and Potential, B. North-Holland Mathematics Studies, vol. 72 (1982). Translated from the French by J.P. Wilson
7. Duchon, J., Robert, R.: Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations. Nonlinearity **13**(1), 249–255 (2000)
8. Flandoli, F., Gątarek, D.: Martingale and stationary solutions for stochastic Navier–Stokes equations. Probab. Theory Relat. Fields **102**(3), 367–391 (1995)
9. Flandoli, F., Gubinelli, M., Hairer, M., Romito, M.: Rigorous remarks about scaling laws in turbulent fluids. Commun. Math. Phys. **278**(1), 1–29 (2008)
10. Flandoli, F., Romito, M.: Markov selections and their regularity for the three-dimensional stochastic Navier–Stokes equations. C. R. Math. Acad. Sci. Paris **343**(1), 47–50 (2006)
11. Flandoli, F., Romito, M.: Markov selections for the three-dimensional stochastic Navier–Stokes equations. Probab. Theory Relat. Fields **140**(3–4), 407–458 (2008)
12. Flandoli, F., Romito, M.: Regularity of transition semigroups associated to a 3D stochastic Navier–Stokes equation. In: Baxendale, P.H., Lototski, S.V. (eds.) Stochastic Differential Equations: Theory and Application. Interdisciplinary Mathematical Sciences, vol. 2. World Scientific, Singapore (2007)
13. Flandoli, F., Romito, M.: In preparation (2008)
14. Frisch, U.: Turbulence: The Legacy of A. N. Kolmogorov. Cambridge University Press, Cambridge (1995)
15. Goldys, B., Maslowski, B.: Exponential ergodicity for stochastic Burgers and 2D Navier–Stokes equations. J. Funct. Anal. **226**(1), 230–255 (2005)
16. Meyn, S.P., Tweedie, R.L.: Markov Chains and Stochastic Stability. Communications and Control Engineering Series. Springer, London (1993)
17. Odasso, C.: Spatial smoothness of the stationary solutions of the 3D Navier–Stokes equations. Electron. J. Probab. **11**(27), 1083–6489 (2006) (electronic)
18. Odasso, C.: Exponential mixing for the 3D stochastic Navier–Stokes equations. Commun. Math. Phys. **270**(1), 109–139 (2007) (electronic)
19. Romito, M.: Existence of martingale and stationary suitable weak solutions for a stochastic Navier–Stokes system. Preprint (2001). Available at http://arXiv.org/abs/math.PR/0609318
20. Stroock, D.W., Varadhan, S.R.S.: Multidimensional diffusion processes. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 233. Springer, Berlin (1979)
21. Temam, R.: Navier–Stokes Equations and Nonlinear Functional Analysis. 2nd edn. CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 66. SIAM, Philadelphia (1995)